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The Diurnal Variation of Terrestrial Magnetism

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IV. *The Diurnal Variation of Terrestrial Magnetism.*

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1. In a previous communication* I proved that the Diurnal Variation of Terrestrial Magnetism has its origin outside the earth's surface and drew the natural conclusion that it was caused by electric currents circulating in the upper regions of the atmosphere. If we endeavour to carry the investigation a step further and enquire into the probable origin of these currents, we have at present no alternative to the theory first proposed by Balfour Stewart that the necessary electromotive forces are supplied by the permanent forces of terrestrial magnetism acting on the bodily motion of masses of conducting air which cut through its lines of force. In the language of modern electrodynamics the periodic magnetic disturbance is due to Foucault currents induced in an oscillating atmosphere by the vertical magnetic force. The problem to be solved in the first instance is the specification of the internal motion of a conducting shell of air, which shall, under the action of given magnetic forces, determine the electric currents producing known electromagnetic effects. Treating the diurnal and semidiurnal variations separately, the calculation leads to the interesting results that each of them is caused by an oscillation of the atmosphere which is of the same nature as that which causes the diurnal changes of barometric pressure. The phases of the barometric and magnetic oscillations agree to about $1\frac{3}{4}$ hours, and it is doubtful whether this difference may not be due to uncertainties in the experimental data. In the previous communication referred to I already tentatively suggested a connexion between the barometric and magnetic changes, but it is only recently that I have examined the matter more closely. In the investigation which follows I begin by considering the possibility that both variations are due to one and the same general oscillation of the atmosphere. The problem is then absolutely determined if the barometric change is known, and we may calculate within certain limits the conducting power of the air which is sufficient and necessary to produce the observed magnetic effects; this conducting power is found to be considerable. It is to be observed, however, that the electric currents producing the magnetic variations circulate only in the upper layers of the atmosphere, where the pressure is too small to affect the barometer; the two

* 'Phil. Trans.,' A, vol. 180, p. 467 (1889).

variations have their origin therefore in different layers, which may to some extent oscillate independently. Though we shall find that the facts may be reconciled with the simpler supposition of one united oscillation of the whole shell of air, there are certain difficulties which are most easily explained by assuming possible differences in phase and amplitude between the upper and lower layers. If the two oscillations are quite independent, the conducting power depending on the now unknown amplitude of the periodic motion cannot be calculated, but must still be large, unless the amplitude reaches a higher order of magnitude than we have any reason to assume.

The mathematical analysis is simple so long as we take the electric conductivity of the air to be uniform and constant; but the great ionisation which the theory demands requires some explanation, and solar radiation suggests itself as a possible cause. Hence we might expect an increased conducting power in summer and in daytime as compared with that found during winter and at night. Observation shows, indeed, that the amplitude of the magnetic variation is considerably greater in summer than in winter, and we know that the needle is at comparative rest during the night. The variable conducting power depending on the position of the sun helps us also to overcome a difficulty which at first sight would appear to exclude the possibility of any close connexion between the barometric and magnetic variations; the difficulty is presented by the fact that the change in atmospheric pressure is mainly semidiurnal, while the greater portion of the magnetic change is diurnal. This may to some extent be explained by the mathematical calculation, which shows that the flow of air giving a 24-hourly variation of barometric pressure is more effective in causing a magnetic variation than the corresponding 12-hourly variation, but the whole difference cannot be accounted for in this manner. If, however, the conductivity of air is greater during the day than during the night, it may be proved that the 12-hourly variation of the barometer produces an appreciable periodicity of 24 hours in the magnetic change, while there is no sensible increase in the 12-hourly magnetic change due to the 24-hourly period of the barometer. The complete solution of the mathematical problem for the case of a conducting power proportional to the cosine of the angle of incidence of the sun's rays is given in Part II. But even this extension of the theory is insufficient to explain entirely the observed increased amplitude of the magnetic variation during summer. We are, therefore, driven to assume either that the atmospheric oscillation of the upper layer is greater in summer than in winter and is to that extent independent of the oscillation of the lower layers, or that the ionising power of solar radiation is in some degree accumulative and that atmospheric conductivity is therefore not completely determined by the position of the sun at the time. The increased amplitude at times when sunspots are frequent is explained by an increased conductivity corresponding to an increase in solar activity. All indications, therefore, point to the sun as the source of ionisation, and ultra-violet radiation seems to be the most plausible cause.

2. The velocity potential of a horizontal irrotational motion of the earth's atmosphere considered as an infinitely thin shell is necessarily expressible as a series of spherical harmonics from which we may select for consideration the one of degree n and type σ, writing it $\psi_n{}^\sigma \sin\{\sigma(\lambda+t)-\alpha\}$ or $\psi_n{}^\sigma$ according to convenience. The longitude λ is measured from any selected meridian towards the east, while t is the time of the standard meridian in angular measure. As in the greater part of the investigation λ and t occur in the combination $\lambda+t$ only, we may frequently omit t without detriment to the clearness, noting, however, that if the differentiation with respect to t is replaced by a differentiation with respect to λ we must apply the factor $2\pi/\mathrm{N}$, where N is the number of seconds in the day.

I consider in the first instance the electric currents which are induced in air moving horizontally under the influence of the earth's vertical magnetic force. Assuming the earth to be a uniformly magnetised sphere, its potential may be resolved into the zonal harmonic of the first degree and the tesseral harmonic of the first type and degree. The angle between the magnetic axis and the geographical axis not being great, the zonal harmonic constitutes by far the largest part, and forms the first subject of our investigation. As far as this part is concerned, we may put the vertical force equal to $\mathrm{C}\cos\theta$, where θ is the colatitude and C, measured upwards, has a numerical value differing little from $-\frac{2}{3}$.

The components of electric force, X and Y, measured towards the south and east respectively, are

$$\mathrm{X}a = \mathrm{C}\cot\theta\frac{d\psi}{d\lambda}, \quad \mathrm{Y}a = -\mathrm{C}\cos\theta\frac{d\psi}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and these equations may be written in the form

$$\left.\begin{aligned} n\,.\,n+1\,.\,\mathrm{X}a &= \mathrm{C}\frac{d^2\psi}{d\theta\,d\lambda} + \mathrm{C}\frac{d}{\sin\theta\,d\lambda}\left(n\,.\,n+1\,.\,\psi\cos\theta-\sin\theta\frac{d\psi}{d\theta}\right) \\ n\,.\,n+1\,.\,\mathrm{Y}a &= \mathrm{C}\frac{d^2\psi}{\sin\theta\,d\lambda^2} - \mathrm{C}\frac{d}{d\theta}\left(n\,.\,n+1\,.\,\psi\cos\theta-\sin\theta\frac{d\psi}{d\theta}\right) \end{aligned}\right\} \quad . \quad . \quad (2),$$

where n is the degree of the harmonic.

The identity between (1) and (2) is obvious as regards the first of the equations, and the reduction of the second is obtained with the help of the fundamental equation

$$\sin\theta\frac{d}{d\theta}\sin\theta\frac{d\psi}{d\theta}+\frac{d^2\psi}{d\lambda^2}+n\,.\,n+1\,.\,\sin^2\theta\,.\,\psi = 0,$$

ψ being a zonal harmonic of degree n.

The components of electric force are by (2) reduced to the form

$$\mathrm{X} = -\frac{d\mathrm{S}}{a\,d\theta}+\frac{d\mathrm{R}}{a\sin\theta\,d\lambda}; \quad \mathrm{Y} = -\frac{d\mathrm{S}}{a\sin\theta\,d\lambda}-\frac{d\mathrm{R}}{a\,d\theta} \quad . \quad . \quad . \quad . \quad (3),$$

and may be divided into two portions, the first of which is derivable from a potential

$$S = -C\frac{d\psi}{d\lambda}\Big/ n\,.\,(n+1).$$

In the steady state this part is balanced by a static distribution of electricity revolving round the earth and causing a variation in the electrostatic potential which is found to be too weak to affect our instruments. The second portion of the electric force produces electric currents; these, neglecting electric inertia—which will be considered later—have ρR as current function, where ρ is the conductivity of the medium.

The comparison of (2) and (3) shows that

$$n\,.\,n+1\,.\,R = C\left(n\,.\,n+1\,.\,\psi\cos\theta - \frac{d\psi}{d\theta}\sin\theta\right),$$

and by means of well known reductions R may be expressed in the normal form

$$(2n+1)\,n\,.\,n+1\,.\,R = C\left[n^2(n-\sigma+1)\,\psi_{n+1} + (n+1)^2(n+\sigma)\,\psi_{n-1}\right]. \quad . \quad . \quad (4).$$

Here ψ_{n+1} and ψ_{n-1} are the two spherical harmonics of degree n and type σ which have the same numerical factor as the current function ψ_n.

I shall confine myself to the two principal portions of the diurnal variation of barometric pressure which are associated with the velocity potentials

$$\psi_1{}^1 = A_1\sin\theta\sin\{(\lambda+t)-\alpha_1\} \quad \text{and} \quad \psi_2{}^2 = 3A_2\sin^2\theta\sin\{2(\lambda+t)-\alpha_2\}.$$

The corresponding electric current functions are seen by (4) to be

$$\rho R_2{}^1 = \tfrac{1}{6}\rho C A_1\psi_2{}^1 \quad \text{and} \quad \rho R_3{}^2 = \tfrac{2}{15}\rho C A_2\psi_3{}^2\,. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

It is shown by CLERK MAXWELL ('Electricity and Magnetism,' vol. II., p. 281) that the magnetic forces accompanying the currents in spherical sheets which are derivable from a current function having the form of a surface harmonic are obtained from a magnetic potential which is equal to the same harmonic multiplied by a factor which inside the spherical shell is $-4\pi(n+1)\,r^n/(2n+1)\,a^n$. The thickness of the atmosphere being negligible compared with the radius of the earth, we may put $r = a$, and obtain thus, for the magnetic potential Ω due to the induced electric currents,

$$-\Omega = \left[\tfrac{2}{5}A_1\psi_2{}^1\sin\{(\lambda+t)-\alpha_1\} + \tfrac{32}{105}A_2\psi_3{}^2\sin\{2(\lambda+t)-\alpha_2\}\right]\pi\rho e C \quad . \quad . \quad (6).$$

The quantity e represents the thickness of the shell of the conducting layer, and is introduced because the current functions used above yield current densities, while MAXWELL'S result applies to functions which lead directly to currents. Our equation (6) represents the potential of the diurnal variation of terrestrial magnetism calculated from an atmospheric oscillation according to our theory, and agrees in form with the principal terms of that variation as observed when the average annual value is considered and the seasonal changes are disregarded.

3. It remains to be seen whether the calculated variation agrees as regards phase and can be made to coincide in magnitude by a reasonable value of the conductivity and thickness of the effective layers of the atmosphere.

For this purpose we first obtain a value for the constants A_1 and A_2. If δp be the variation of the pressure p, and $d\sigma$ the corresponding change of the density σ, we have

$$\frac{\delta p}{p} = \frac{d\sigma}{\sigma} = -\frac{d\psi}{v^2 dt},$$

where ψ is the velocity potential and v the velocity of sound. Under the assumption that the whole atmosphere oscillates equally in all its layers, $\delta p/p$ will be the same at every point of a vertical line, and we may, therefore, determine its value at the surface of the earth.

According to HANN ('Meteorologie,' p. 189), the diurnal change of the barometer at the equator, measured in millimetres, is represented by

$$0{\cdot}3 \sin(\lambda+t) + 0{\cdot}92 \sin\{2(\lambda+t) + 156^\circ\}.$$

If this expression be denoted by δp, we must assign the value of 760 to p to bring the units into harmony.

It follows that at the equator

$$\psi = [-0{\cdot}3 \cos(\lambda+t) + 0{\cdot}46 \cos\{2(\lambda+t) + 156^\circ\}]\,Nv^2/2\pi p \quad . \quad . \quad . \quad (7).$$

The numerical value of $Nv^2/2p$ is $6{\cdot}281 \times 10^{10}$ ($N = 86400$; $v^2 = 11{\cdot}05 \times 10^8$), or $98{\cdot}5a$, where a is the radius of the earth.

The constants A_1 and A_2 of the velocity potential in (6) are, therefore, determined by

$$\pi A_1 = 0{\cdot}3 \times 98{\cdot}5a = 29{\cdot}6a, \quad \text{and} \quad \pi A_2 = 0{\cdot}153 \times 98{\cdot}5a = 15{\cdot}1a \quad . \quad . \quad (8).$$

We ultimately get for the calculated magnetic potential

$$\Omega/a = [11{\cdot}8 \cos(\lambda+t) - 4{\cdot}6 \cos 2(\lambda+t-102^\circ)]\,\rho e C,$$

or, introducing the value of C and restoring the term containing the latitude,

$$\Omega/a = [7{\cdot}89\psi'_2 \cos(\lambda+t-180^\circ) + 3{\cdot}07\psi_3^2 \cos 2(\lambda+t-102^\circ)]\,\rho e \quad . \quad . \quad (9).$$

The principal terms of the diurnal and semidiurnal variations of magnetic force, abstraction being made of seasonal changes, were found in my previous communication to be

$$10^6\Omega/a = 89\psi_2' \cos(\lambda+t-156^\circ) + 11{\cdot}16\psi_3^2 \cos 2(\lambda+t-74{\cdot}5^\circ) \quad . \quad . \quad (10).$$

If we compare the phases, we find that the magnetic potential calculated from the barometric variation lags behind the observed one by an amount which is 1 hour and 36 minutes for the diurnal and 1 hour and 50 minutes for the semidiurnal variation, showing a remarkably close agreement in the two terms. As regards amplitude, we can establish agreement by adjusting the value of ρe, but the same value ought to satisfy both terms, which is not the case, ρe being $3{\cdot}63 \times 10^{-6}$ if calculated from the 12-hourly variation, and $11{\cdot}3 \times 10^{-6}$ if calculated from the 24-hourly variation.

4. The tacit assumption has been made that the barometric variation is distributed over the surface of the earth according to the simplest harmonic term consistent with each period, so that for the diurnal variation the amplitude would be proportional to the cosine of the latitude, and for the semidiurnal variation to the square of the cosine.

The experimental data show, however, that there are other terms to be considered, and for the semidiurnal variation Dr. ADOLF SCHMIDT has obtained the best agreement by introducing the harmonic of the fourth degree, having an amplitude which at the equator would amount to the twelfth part of the whole effect. The amplitude of the second term in Equation 9 would consequently be reduced to $3{\cdot}07 \times \frac{11}{12} = 2{\cdot}814$, and a higher harmonic would be introduced; but the experimental evaluation of these higher terms in the magnetic variation is too uncertain to be taken account of at present, their effect in any case being small. As regards the 24-hourly variation, its dependence on latitude has not been clearly established. The term, as commonly observed, is much affected by local circumstances, and HANN takes therefore the observation on board ship to represent the true phenomenon so far as it depends on the atmosphere as a whole. While greater consistency is thus gained, the observations on board ship cannot lay claim to the same accuracy as those taken on land, and, as the figures show, considerable uncertainty still prevails. In the following table the values given by HANN are collected together :—

Latitude.	Amplitude in millimetres.	Latitude.	Amplitude in millimetres.
°		°	
4·5	0·262	33·8	0·148
11·1	0·265	35·9	0·140
15·8	0·268	37	0·342
23	0·115	40·7	1·85

These numbers do not follow any very simple law and can only be very partially represented by an expression varying as the cosine of the latitude. The rapid increase in the amplitude at latitudes of about 40° suggests the presence of the third harmonic, and treating the figures by the method of least squares we are led to an expression

$$\delta p = 0{\cdot}49 \sin\theta - 0{\cdot}33 \left\{\tfrac{3}{2} \sin\theta \left(5 \cos^2\theta - 1\right)\right\}.$$

If this equation were to represent correctly the distribution in latitude of the diurnal term, the calculated amplitude of the magnetic potential would be increased considerably more than is required to bring it into harmony with the semidiurnal term, because not only is the amplitude of the term of the barometric variation in $\sin\theta$ increased, but the additional term gives rise to a magnetic potential which is of the same degree and type and more than doubles the effect of the first term.

Very little importance, however, can be attached to this calculation, which depends to a great extent on the last entry of the foregoing table; but enough has been said to show that our present knowledge of the 24-hourly variation of the barometric pressure is very uncertain, and that a term of the third degree in its expression is likely to diminish materially the discrepancy between the electric conductivity of the atmosphere as derived from the diurnal and semidiurnal periods.

5. We must next turn our attention to several corrections which modify the calculated values without, however, introducing material changes. The observed magnetic variations have been treated as if they were wholly due to outside causes, although it was shown in my previous communication that an appreciable portion of it was an effect of electric currents induced inside the earth by the varying potential itself. What we observe is the resultant of the original outside effect and its concomitant induced inside effect. To explain the absence of time lag of the induced variation it was necessary to assume a good conductivity of an inner core and small conductivity of the outer shell. An estimate may be made of the radius of the conducting core. If the outer potential is represented by $\Omega r^n a^{-n}$, where a is the radius of the earth, and the inner potential is $\kappa\Omega r^{-n-1}a^{n+1}$, an estimate of κ may be obtained by the fact that if the inner conductivity is sufficiently great the vertical force is entirely destroyed at the surface of the inner core. If this has a radius r_0, it follows that $nr_0^n a^{-n}$ and $(n+1)\kappa r_0^{-n-1}a^{n+1}$ must have equal values, or that $\kappa = nr_0^{2n+1}/(n+1)\,a^{2n+1}$.

The resultant potential at the surface of the earth is, therefore,

$$\Omega\left[1+\frac{n}{n+1}\left(\frac{r_0}{a}\right)^{2n+1}\right].$$

If this expression is multiplied by n, we obtain the vertical force calculated from the horizontal force on the supposition that the whole effect comes from the outside.

The *observed* vertical force, on the other hand, is

$$n\Omega\left[1-(r_0/a)^{2n+1}\right].$$

The previous result showed that the actual vertical force was about half the calculated one, the principal term being that due to $n = 2$. We find in this way $(r_0/a)^5$.

The thickness of the outer non-conducting crust would thus appear to be about 1000 kilometres, and cannot, therefore, be connected with the layer having a thickness of about 30 miles which shows itself in its effect on seismic waves, and, according to STRUTT, contains the radioactive matter. On the other hand, it is quite likely that the outer shell is identical with that which the discussion of the propagation of seismic waves shows to have different elastic properties from the nucleus, and which, according to WIECHERT's recent researches, has a thickness of 1500 kilometres.

The observations show that the internal potential has a value equal to one-fourth of the external one, or that the external potential represents 0·8 of the whole. For $n = 3$, using the same value of r_0, we similarly find that the outside effect is 0·84 times that of the whole. The coefficients in equation (9) should therefore be diminished by multiplying with 0·80 and 0·84 respectively.

6. We may now complete the investigation as far as it relates to uniform conductivities. The magnetic and geographical poles of the earth not coinciding, the vertical force is not simply proportional to the cosine of the colatitude, but a term must be added proportional to $\sin\theta\cos\lambda$, where λ is measured from the meridian 68° 31′ west of Greenwich, which is that containing the magnetic axis. I discuss the effect of the inclination of the magnetic axis somewhat in detail, as it will give us a good test of the proposed theory when suitable observations will be available. Leaving out the factor $C\tan\phi$, where C represents the vertical force at the geographical pole and ϕ the colatitude of the magnetic pole, the electric forces, as far as they concern us at present, are

$$X = \cos\lambda\frac{d\psi}{d\lambda}; \quad Y = -\sin\theta\cos\lambda\frac{d\psi}{d\theta}.$$

If these values be substituted in (3), the elimination of S leads to the equation

$$\frac{d}{d\lambda}\cos\lambda\frac{d\psi}{d\lambda} + \frac{d}{d\theta}\sin^2\theta\cos\lambda\frac{d\psi}{d\theta} = \frac{d^2R}{\sin\theta\, d\lambda^2} + \frac{d}{d\theta}\sin\theta\frac{dR}{d\theta}. \quad . \quad . \quad (11).$$

It will be shown in Part II. that

$$\left(\frac{d}{d\lambda}\cos\lambda\frac{d}{d\lambda} + \cos\lambda\frac{d}{d\theta}\sin^2\theta\frac{d}{d\theta}\right)\psi_n^\sigma\sin\sigma(\lambda+t-\alpha)$$

is equal to

$$\tfrac{1}{2}\frac{\sin\theta}{2n+1}\left[(a_1\psi_{n-1}^{\sigma+1}+b_1\psi_{n+1}^{\sigma+1})\sin\{\sigma(\lambda+t)+\lambda-\alpha\}\right.$$
$$\left.+(a_2\psi_{n-1}^{\sigma-1}+b_2\psi_{n+1}^{\sigma-1})\sin\{\sigma(\lambda+t)-\lambda-\alpha\}\right],$$

where

$$a_1 = n-1\,.\,n+1; \quad -b_1 = n\,.\,n+2; \quad -a_2 = (n-1)(n+1)(n+\sigma)(n+\sigma-1);$$
$$b_2 = n\,.\,n+2\,.\,(n-\sigma+1)\,.\,(n-\sigma+2).$$

The only case necessary to consider is that in which $n = \sigma$, when $\phi_{n-1}^{\sigma+1} = 0$, so that we may disregard the factor a_1. If $n = \sigma = 1$, then $b_1 = -3$; $a_2 = 0$; $b_2 = 6$; and if $n = \sigma = 2$, then $b_1 = -8$; $a_2 = -36$; $b_2 = 16$.

We can satisfy equation (3) by assuming R to be made up of two or four terms, according as we treat of the diurnal or semidiurnal variations. Remembering that by the fundamental equation

$$\left(\frac{d^2}{\sin\theta\, d\lambda^2}+\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}+n\,.\,n+1\right)\psi_n^{\sigma}=0,$$

we find the terms which are introduced by the inclination of the magnetic axis to be for the diurnal variation

$$\tfrac{1}{12}R_2^{2}\sin(2\lambda+t-\alpha)-\tfrac{1}{6}R_3^{0}\sin(t-\alpha),$$

and for the semidiurnal variation

$$\tfrac{1}{15}R_3^{3}\sin(3\lambda+2t-\alpha)+\tfrac{9}{5}R'_1-\tfrac{2}{15}R'_3)\sin(\lambda+2t-\alpha).$$

The additional terms are therefore, restoring the constant factor,

$$-\Omega=-[\tfrac{1}{5}A_1\psi_2^{2}\sin(t'+\lambda-\alpha_1)-\tfrac{2}{5}A_1\psi'_2\sin(t'-\lambda-\alpha_1)+\tfrac{16}{5}A_2\psi_3^{3}\sin(2t'+\lambda-\alpha_2)$$
$$+(\tfrac{24}{5}\psi'_1-\tfrac{32}{105}\psi'_3)A_2(\sin 2t'-\lambda-\alpha_2)]\,\pi\rho e C\tan\phi \quad . \quad . \quad . \quad . \quad . \quad (12).$$

Here tan ϕ represents the angle between the magnetic and geographical axes ($\tan\phi=0{\cdot}202$), and t' has been introduced to represent the local time $\lambda+t$. The functions ψ are the tesseral functions, so that

$$\psi_2^{0}=\tfrac{3}{2}\cos^2\theta-\tfrac{1}{2},\qquad \psi'_3=\tfrac{3}{2}\sin\theta(5\cos^2\theta-1),\qquad \psi'_1=\sin\theta,$$
$$\psi_2^{2}=3\sin^2\theta,\qquad \psi_3^{3}=15\sin^3\theta.$$

Equations (12) show that if the inclination of the magnetic axis be taken into account, the diurnal variations do not entirely depend on local time. A barometric variation of a certain period is accompanied by a magnetic variation of the same period, as is obvious; but if the lines of equal magnetic potential in the diurnal variation are drawn as in my previous communication, a barometric variation represented by $\psi_n^{\sigma}\cos\sigma t$ results in a magnetic potential containing terms

$$\psi_{n-1}^{\sigma-1}\cos(\sigma-1)\lambda;\quad \psi_{n+1}^{\sigma-1}\cos(\sigma-1)\lambda;\quad \psi_{n-1}^{\sigma+1}\cos(\sigma+1);\quad \psi_{n+1}^{\sigma+1}\cos(\sigma+1)\lambda.$$

These equipotential lines and their coincident stream lines revolve with velocities $\sigma\omega/\sigma-1$ and $\sigma\omega/(\sigma+1)$ round the earth, ω being its angular velocity, and in this way variations proportional to $\cos\sigma t$ are produced.

In order to estimate the magnitude of these terms, consider the diurnal variation, the normal term of which has been found to be equal to $\tfrac{2}{5}A_1\psi'_2\cos(t-\alpha)$.

Along a meridian circle for which λ is either 0 or π, the additional terms are, putting A_1 equal to unity, for $\lambda=0$

$$[\tfrac{3}{10}\sin^2\theta-\tfrac{1}{10}(3\cos^2\theta-1)]\cos(t-\alpha)=\tfrac{1}{10}(1-3\cos 2\theta)\cos(t-\alpha),$$

and for $\lambda = \pi$

$$[\tfrac{3}{10} \sin^2 \theta + \tfrac{1}{2} (3 \cos^2 \theta - 1)] \cos (t-\alpha) = \tfrac{1}{5} \cos (t-\alpha).$$

In these equations the numerical value 0·2 has been introduced for tan ϕ.

At the equator the additional terms, therefore, have amplitudes $\frac{2}{5}$ and $\frac{1}{5}$ respectively, as compared with the main diurnal term. The force to geographical west being proportional to the potential, we may take these numbers to be the amplitude of the westerly force variations. The main variation is proportional to sin θ cos θ, and has zero value at the equator in the tropical region. The additional terms are therefore the ruling terms at the equator. The horizontal force along the same circle has unit amplitude, measured on the same scale, so that the new terms come well within the range of our observational powers. It would be interesting to trace them, but it should be remarked that only observations made near the equinox are suitable for the purpose, as the seasonal terms, which yet remain to be discussed, would otherwise interfere.

7. We may interrupt the progress of our investigation for a moment to inquire into the magnitude of the electrostatic effect dependent on the potential S which was found equal to $-C \frac{d\psi}{d\lambda}\Big/ n \,.\, n+1$, leading to a vertical electric force $C\sigma\psi/(n+1)\,a$. In the two cases which concern us, $\sigma = n$, and ψ has values at the equator which were found to be $30a/\pi$ and $15\,a/\pi$ respectively. It follows that the variation of vertical electric force is of the order of 3 C.G.S. units, which is only 1 volt per 300 kilometres. This may be disregarded.

8. The previous discussion has only taken the earth's vertical magnetic force into consideration. The horizontal force causes, in combination with a horizontal atmospheric oscillation, a vertical electromotive force, and so far as this produces electric currents, their flow is in opposite directions in strata which are vertically above each other. The magnetic effect is therefore of a smaller order of magnitude than that due to vertical force.

9. In calculating the currents from the electric forces, I have applied OHM's law, and therefore neglected the effects of electric inertia; but it is not difficult to estimate the change of phase which results from self-induction. Using the equations given by MAXWELL* for spherical current sheets, we find that if R is the function defined by equation (3), and ϕ the current function,

$$\phi + L\rho \frac{d\phi}{dt} = R\rho,$$

where ρ is the conductivity, and $L = (2n+1)/4\pi a$; provided that R is a surface harmonic of degree n. If the latter function is proportional to cos κt, we find in the usual way

$$\phi = \rho \cos \epsilon \cos (\kappa t - \epsilon), \quad \tan \epsilon = \frac{4\pi a \kappa \rho}{2n+1}.$$

* See CLERK MAXWELL, 'Electricity and Magnetism,' vol. II., § 672.

If we take the current sheets to be of finite thickness e, and ρ denotes conductivity referred to unit volume, we must write

$$\tan \epsilon = \frac{4\pi a \kappa \rho e}{2n+1}.$$

If ρe has the value previously determined by the semidiurnal variation, I calculate a retardation of 1 hour for the semidiurnal and about $1\frac{1}{2}$ hours for the diurnal term. The amplitude would be reduced by about 14 per cent. There are various causes, notably the inequalities of conductivity, tending to diminish the retardation, so that we may consider that self-induction would not cause a shift of phase amounting to more than an hour, but it is in the opposite direction to that indicated by the observations, if the barometric and magnetic oscillations are due to identical causes.

10. It is known that the air contains an excess of positive electricity, and the question might be raised whether the oscillations of the atmosphere do not convectively produce direct magnetic effects. If E be the total quantity of electricity contained in a vertical column of unit cross section, and V, the velocity of air in that column, supposed to be uniform, the total current in the atmosphere across a vertical area of unit width is EV, and the magnetic force at the surface of the earth is of the order of magnitude 2πEV. The quantity E must be equal and opposite to the surface electrification of the earth, which itself is equal to $F/4\pi v^2$, where v is the velocity of light, and F the normal fall of potential, which we may put equal to 1 volt per centimetre, or to 10^8. The magnetic force has therefore a magnitude of order $5{\cdot}8V \times 10^{-14}$. If the velocity potential of the atmospheric oscillation is $A\psi_n^\sigma \sin \sigma(\lambda+t)$, the velocity in the two cases considered is greatest at the equator, where its maximum rises to $\sigma A\psi_n^\sigma/a$, which for the diurnal and semidiurnal change is A_1/a and $6A_2/a$ respectively. It follows from the numerical values given in (8) that the maximum equatorial velocities are 10 and 30 centims. per second respectively. The magnetic forces due to such velocities are quite insignificant. In the literature referring to the subject we frequently find it suggested that magnetic disturbances are due to moving masses of electrified air, some writers even going so far as to say that this has been *proved*; it may be demonstrated, on the contrary, that the assumed cause is insufficient. For horizontal air currents this has just been demonstrated, and the effects of ascending or descending currents are still less efficient. If a column of air of cylindrical shape having as base a circle of radius r rises or falls with velocity V, and it is imagined as an extreme case that the column extends indefinitely in both directions, the magnetic force at the boundary is $2\pi r$EV, where E is the electric density. At the surface of the earth the ionisation is such that the free electric charges of each kind amount to about 1 electrostatic unit per cubic metre. Let us assume one kind to be suppressed altogether, so that this number represents the electric volume density, or in electromagnetic measure $0{\cdot}33 \times 10^{-16}$. If r be 1 kilometre, and the velocity that of an express train, or 30 metres a second, the

magnetic effect would be $6 \cdot 3 \times 10^{-8}$ C.G.S. This is insignificant and leaves a good margin for a greater sectional area of the ascending current, especially if it is remembered that both our assumed velocity and the volume charges are many times greater than is allowable. Magnetic effects due to the motion of electrified air must therefore be ruled out as effective causes of either regular or irregular magnetic changes.

11. The daily variation of the magnetic forces includes a strong seasonal term, the amplitudes being greater in summer than in winter. In order to explain this term according to the theory advocated, it is necessary to assume a greater electric conductivity of the atmosphere in summer than in winter, or an oscillation of greater amplitude, which is not, however, indicated by the barometric changes. That the conductivity depends on the position of the sun, and may therefore vary with the season, is suggested by the relation in phase between the diurnal and semidiurnal terms, these terms combining together so as to leave the needle comparatively quiescent during the night. Reserving the possible causes of the conductivity and its dependence on solar position for further discussion, we may complete the theoretical investigation by introducing a variable conductivity. The simplest supposition to make will be that the conducting power in any small volume is proportional to the cosine of the angle between the vertical and the line drawn to the sun, or, in other words, proportional to the cosine of the angle, measured at the centre of the earth on the celestial sphere, between the sun and the small volume considered. This angle (ω) is expressed in spherical co-ordinates by

$$\cos \omega = \sin \delta \cos \theta + \sin \theta \cos \delta \cos \lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

where λ is the longitude measured from the meridian passing through the sun, and θ is measured from the pole, δ representing the sun's declination. To put the assumed law of conductivity into mathematical form, we write

$$\rho = \rho_0 + \rho_1 \cos \omega.$$

If $\rho_1 = \rho_0$, the conductivity would be zero at a point opposed to the sun, and this is the highest admissible value of ρ_1. In order to keep our investigation as general as possible, I write

$$\rho = \rho_0 (1 + \gamma' \cos \theta + \gamma \sin \theta \cos \lambda),$$

where γ' and γ may have any assigned values. The solution of our problem is obtained if we can find values of S and R satisfying the equations

$$\left. \begin{aligned} \rho \cot \theta \frac{d\psi}{d\lambda} &= \rho \frac{dS}{d\theta} + \frac{dR}{\sin \theta \, d\lambda} \\ -\rho \cos \theta \frac{d\psi}{d\theta} &= \rho \frac{dS}{\sin \theta \, d\lambda} - \frac{dR}{d\theta} \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

R will now give directly the current function which hitherto was denoted by ρ_0R.

The general problem will be treated in the Appendix, where it is shown that for practical purposes γ and γ' may be treated as small quantities, the squares of which may be neglected. The equations may then be written

$$\left.\begin{aligned} \rho \cot\theta \frac{d\psi}{d\lambda} &= \rho_0 \frac{dS}{d\theta} + (1-\gamma'\cos\theta-\gamma\sin\theta\cos\lambda)\frac{dR}{\sin\theta\, d\lambda} \\ -\rho_0 \cos\theta \frac{d\psi}{d\theta} &= \rho_0 \frac{dS}{\sin\theta\, d\lambda} - (1-\gamma'\cos\theta-\gamma\sin\theta\cos\lambda)\frac{dR}{d\theta} \end{aligned}\right\} \quad . \quad . \quad . \quad (15).$$

Neglecting γ and γ', our previous results give R in terms of ψ. Let Q_n^σ be one part of R thus obtained. The next approximation is found by substituting Q_n^σ for R in the terms of (15), which contain γ and γ'.

The complete value of R, as far as it depends on Q_n^σ, will then be $Q_n^\sigma + R'$, where R' is determined by

$$\left.\begin{aligned} (\gamma'\cos\theta+\gamma\sin\theta\cos\lambda)\frac{dQ_n^\sigma}{\sin\theta\, d\lambda} &= \rho_0 \frac{dS'}{d\theta} + \frac{dR'}{\sin\theta\, d\lambda} \\ -(\gamma'\cos\theta+\gamma\sin\theta\cos\lambda)\frac{dQ_n^\sigma}{d\theta} &= \rho_0 \frac{dS'}{\sin\theta\, d\lambda} - \frac{dR'}{d\theta} \end{aligned}\right\} \quad . \quad . \quad . \quad (16).$$

In the two cases which specially interest us we must substitute for Q_n^σ the values respectively of R_2^1 and ρR_3^2 as determined by (5). The solution of (16) involves the elimination of S'.

Treating the terms containing γ and γ' separately, we find for R' as far as it depends on γ',

$$\gamma'\left(\cot\theta \frac{d^2 Q_n^\sigma}{d\lambda^2} + \frac{d}{d\theta}\sin\theta\cos\theta\frac{dQ_n^\sigma}{d\theta}\right) = -\Sigma n \,.\, n+1 \,.\, \sin\theta\, R'_n,$$

if R' is expressed as a series of harmonics, n being the degree of one of the terms of the series.

The left-hand side may be transformed as shown in the Appendix, the result being given by (25); we obtain in this way

$$\gamma'\left(\frac{(n+2)\,n\,(n-\sigma+1)}{2n+1} Q^\sigma{}_{n+1} + \frac{(n-1)(n+1)(n+\sigma)}{2n+1} Q^\sigma{}_{n-1}\right) = \Sigma n\,(n+1)\, R'_n.$$

R' is therefore expressible by two terms, $R^\sigma{}_{n+1}$ and $R^\sigma{}_{n-1}$, so that

$$(2n+1)(n+1)\,R^\sigma{}_{n+1} = \gamma' n\,(n-\sigma+1)\,Q^\sigma{}_{n+1},$$
$$(2n+1)\,n\,.\,R^\sigma{}_{n-1} = \gamma'(n+1)(n+\sigma)\,Q^\sigma{}_{n-1}.$$

As regards the terms in γ, the elimination of S' leads to

$$\gamma\left(\frac{d}{d\lambda}\cos\lambda\frac{d}{d\lambda} + \cos\lambda\frac{d}{d\theta}\sin^2\theta\frac{d}{d\theta}\right) Q_n^\sigma = -\Sigma n\,(n+1)\sin\theta R'_n.$$

The left-hand side is transformed, as shown in the Appendix, the result being given in (22). We thus find R′ expressible as a sum of four terms—

$$2(2n+1)(n+1)\,R_{n+1}^{\sigma+1} = \gamma n Q_{n+1}^{\sigma+1},$$
$$2(2n+1)\,n\,.\,R_{n-1}^{\sigma+1} = -\gamma(n+1)\,Q_{n-1}^{\sigma+1},$$
$$2(2n+1)(n+1)\,R_{n+1}^{\sigma-1} = -\gamma n(n-\sigma+1)(n-\sigma+2)\,Q_{n+1}^{\sigma-1},$$
$$2(2n+1)\,n(n-1)\,R_{n-1}^{\sigma-1} = \gamma(n-1)(n+1)(n+\sigma)(n+\sigma-1)\,Q_{n-1}^{\sigma-1}.$$

For Q_n^σ we must substitute $\frac{1}{6}CA_1\psi_2^1$ and $\frac{2}{15}CA_2\psi_3^2$ when treating of the diurnal and semidiurnal variations respectively, where ψ_2^1 and ψ_3^2 are the harmonics of type and degree indicated which have the same factors as the current functions ψ_1^1 and ψ_2^2.

To get the magnetic potential, a further multiplication by $-4\pi(n+1)/2n+1$ is required. We see that each barometric variation now leads to six terms in the magnetic potential, the factors of $-\pi\rho_0 eAC\Omega_n^\sigma$ being collected in the following table :—

Diurnal Variation.

Velocity Potential : $A_1\psi_1^1$. Magnetic Potential : $-\pi\rho_0 eA_1C\Sigma B_n^\sigma\Omega_n^\sigma$.

Values of B_n^σ :

	$n = 1.$	$n = 2.$	$n = 3.$
$\sigma = 0$	$\frac{2}{5}\gamma$	—	$-\frac{16}{105}\gamma$
1	$\frac{2}{5}\gamma'$	$\frac{2}{5}$	$\frac{32}{315}\gamma'$
2	—	—	$\frac{8}{315}\gamma$

Semidiurnal Variation.

Velocity Potential : $-\pi\rho_0 eA_2C\Sigma B_n^\sigma\Omega_n^\sigma$. Barometric Variation : $A_2\psi_2^2$.

Values of B_n^σ :—

	$n = 2.$	$n = 3.$	$n = 4.$
$\sigma = 1$	$\frac{64}{105}\gamma$	—	$-\frac{2}{21}\gamma$
2	$\frac{32}{105}\gamma'$	$\frac{32}{105}$	$\frac{4}{63}\gamma'$
3	—	—	$\frac{1}{63}\gamma$

C = Vertical force at geographical North Pole, measured upwards.
ρ_0 = Electric conductivity of atmospheric shell.
e = Thickness of atmospheric shell.

The main terms of the magnetic potentials $\Omega_2^{\,1}$ and $\Omega_3^{\,2}$ are now each affected by both the diurnal and semidiurnal barometric variation, and their relative amplitudes may differ considerably from those calculated on the assumption of a uniform conductivity. If γ has its maximum value, which is unity, we have for these two terms, neglecting an unimportant difference of phase, and leaving out common factors,

$$\Omega_2^{\,1} = \frac{2}{5}A_1 + \frac{64}{105}A_2 = 21\cdot0\,\frac{a}{\pi},$$

$$\Omega_3^{\,2} = \frac{32}{105}A_2 + \frac{8}{315}A_1 = 4\cdot7\,\frac{a}{\pi}.$$

The 24-hourly variation of terrestrial magnetism now takes the lead and as regards westerly force is now 4·7 times as great as the semidiurnal variation, but the latter is still too great for complete agreement with the facts, the observational ratio being 8·8. This remaining discrepancy is not decisive against the accuracy of the assigned cause in view of the uncertainty which attaches to the 24-hourly term in the barometric variation as explained in § 4, and the considerations brought forward in the following paragraph. There is the theoretical possibility of a further increase in the diurnal term through a velocity proportional to $P_2^{\,0}$; the motion specified by this potential would give a barometric oscillation determined by the time of some definite meridian, and though observations seem to indicate the existence of part of the oscillation being of this nature, it is not likely that it is sufficiently great to produce a marked magnetic effect.

12. A general review of the argument, even at the risk of repeating a portion of what has already been said, may be appropriate, and is necessary to show how we are naturally led to the theory here proposed. It will also serve to introduce the consideration of the remaining difficulties and of the possibility of accounting for the amount of ionisation necessary to explain the magnitude of the observed effects.

Our object is to explain the cause of the periodic changes of the terrestrial magnetic forces in so far as they depend on the position of the sun. The diurnal changes may be represented as being governed by a magnetic potential Ω composed of terms of the form $\Omega_n^{\,\sigma}\cos\sigma(\lambda+t)$, where $\Omega_n^{\,\sigma}$ is a surface harmonic; the observed vertical forces show, as proved in my previous communication, that we must seek the cause of the variation outside the earth's surface. Electric currents circulating in our atmosphere and having a current function made up of terms which are respectively proportional to $\Omega_n^{\,\sigma}$ produce the required effect, and we are justified in assuming this—the simplest explanation—to be also the correct one until it is shown to lead to contradictions. The maintenance of the electric currents necessarily requires an electromotive force, and their closed lines of flow dispose of any theory which would seek this force in a static distribution of potential. Electric charges carried along by air currents have been shown to be insufficient to produce appreciable effects, and we are therefore driven to look upon electromagnetic induction as being the only possible cause of the observed effects, the earth's magnetism and atmospheric circulation being the active agents. Assuming as most probable that atmospheric circulation is symmetrical north and south of the equator, the character of the magnetic variation shows that the effective component of terrestrial magnetism has opposite signs in the two hemispheres; it must therefore be the vertical component which is active. We next put the question: What must be the atmospheric circulation which under the action of the vertical magnetic force produces periodical magnetic effects equal to those actually observed? Taking the average for the complete year, the leading terms of the variable magnetic potential are $\Omega_2^1 \cos(\lambda+t)$ and $\Omega_3^2 \cos 2(\lambda+t)$, the amplitude of the diurnal term being equal to about eight times the amplitude of the semidiurnal one. Calculation shows that Ω_2^1 may be produced by a quasi-tidal atmospheric flow having as velocity potential either $\psi_1^1 \cos(\lambda+t)$ or $\psi_3^1 \cos(\lambda+t)$, while the semidiurnal term may be produced by a flow having a velocity potential $\psi_2^2 \cos 2(\lambda+t)$ or $\psi_4^2 \cos 2(\lambda+t)$. But these velocity potentials are exactly what is required for the atmospheric waves causing the daily changes of barometric pressure. The semidiurnal term of the pressure change is the one least affected by local conditions, and its distribution over the earth's surface is therefore accurately known. It is found that ψ_4^2 is small compared with ψ_2^2. As regards the diurnal term having an amplitude at the equator of only one-third of the semidiurnal one, it varies somewhat irregularly and the relative importance of ψ_1^1 and ψ_3^1 is not well ascertained. Assuming the barometric variation to be wholly due to $\psi_1^1 \cos(\lambda+t)$ and $\psi_2^2 \cos 2(\lambda+t)$, we may deduce the magnetic variation and compare it with the observed changes. This has been the course of the investigation in the preceding paragraphs. It is found that the calculated magnetic variations have a phase which lags behind the observed one by about $1\frac{3}{4}$ hours, and this lag is slightly less for the diurnal term, but the difference is insignificant in view of the uncertainties of the data. The amplitude of the calculated diurnal term is about $2\frac{1}{2}$ times as great as that of the semidiurnal one, while observation gives, as has already been stated, a ratio of 8 for the two terms. But if part of the barometric variation is due to a term ψ_3^1—and there is some evidence that this is the case—agreement in the ratio of the two terms may be secured. There is, however, a further cause tending to increase the semidiurnal magnetic variation. In order to explain, on the basis of our theory, the difference in the magnetic changes between summer and winter, we must assume that the conductivity of the atmosphere is greater in that hemisphere which is more directly under the influence of the solar rays. Assuming that the electric conductivity is proportional to $1 + \cos\omega$, where ω is the angle measured on the celestial sphere between the sun and the point considered, the calculated semidiurnal term reaches a value which is 4·7 times as great as that of the diurnal term, so that the term ψ_3^1 is now called upon to a much smaller extent for making up the deficiency in the diurnal term. The supposed inequalities of conductivity, though helping towards a better agreement between the diurnal and semidiurnal terms, are insufficient to account completely for the large excess of the summer variation over that observed in winter. This inequality is expressed by Ω_3^2 for the semidiurnal variation and Ω_2^1 for the diurnal variation, and its relative magnitude is indicated by the ratios Ω_3^2/Ω_2^2 and Ω_2^1/Ω_1^1 respectively. The calculated value of both ratios is shown by the tables in § 10 to be $\gamma' = \sin \delta$, where δ is the sun's declination. If we compare the variations during the six summer months with those during the six winter months, we must substitute for $\sin \delta$ its average value, which is about 0·26. On the other hand, the results of my previous communication allow us to deduce the ratios Ω_3^2/Ω_2^2 and Ω_2^1/Ω_1^1 from the observations, and we find in this 0·6 and 0·8 respectively, or values between two and three times as great as those calculated from the assumed law of conductivity.

To explain the difference we might imagine some cumulative effect, so that in midsummer the conduction would be greater than in winter even for the same elevation of the sun, but our present knowledge does not justify us in assuming this to be the case. I am inclined, therefore, to consider that the cause of the discrepancy lies in the fact that, as already suggested in the introductory remarks, the oscillations responsible for the barometric and magnetic phenomena are to some extent independent of each other, affecting different layers of the atmosphere. There are theoretical reasons why this should be so. It is now, I think, generally recognised that the importance of the semidiurnal variation of the barometer is due to the fact that the free period of the atmospheric oscillation, dependent on the velocity potential ψ_2^2, is very nearly equal to 12 hours. But it is to be remarked that if concentric layers of the atmosphere be considered separately, there must be a considerable variation in the free periods owing to differences of temperature, and in the highest regions, in which alone electric currents of sufficient intensity can circulate, the temperature is probably so low that the free periods would be more than doubled. If we take these highest layers to oscillate to some extent independently, we should not therefore find the semidiurnal variation stand out in the same way as it does for the lower layers. Further, the inequalities of solar radiation in the two hemispheres near solstice ought to cause an appreciable oscillation dependent on the velocity potentials ψ'_2 and ψ_3^2. The barometric variation due to ψ_3^2 is unimportant compared with that due to ψ_2^2, because although the *forced* period is 12 hours, the free period corresponding to the motion involved in it has now a different value; but in the upper layers the relative importance of ψ_3^2 would be increased, or, as it would be more correct to say, the relative importance of ψ_2^2 disappears. This would account for the magnitude of the seasonal term in the magnetic variation.

The suggested partial independence of the oscillations of the upper and lower atmospheres may also explain the discrepancy of phase, which we found to be $1\frac{3}{4}$ hours, but is in reality somewhat greater, owing to the fact that self-induction has been neglected in calculating the phase. With the calculated conductivity, self-induction would cause a retardation of about one hour if the amplitude of oscillation is that deduced from the barometric variation. If the amplitude in the conducting regions is greater, the effect of self-induction is correspondingly less, because a smaller conductivity would then be required to account for the magnetic change.

A few words should be said on the uncertainties of the data which serve as a test of the proposed theory and which are derived from my previous calculation of the variation potential. In deducing that potential I was practically obliged to confine myself to the records of four observatories (Bombay, Lisbon, Greenwich, St. Petersburg), all four being situated in the northern hemisphere; and the year 1870 being the only one for which records were available at all four stations, I had to base my calculations on the figures for that year. Unfortunately, 1870 was a year of unusual sunspot activity and the magnetic records for that year cannot be taken as quite normal. It is probable that if the average of a number of years were taken, the phases of the components and their relative amplitudes might be somewhat altered; but I do not think that, as far as the averages for the whole year are concerned, the results of the present investigation would be materially altered. A renewed discussion is, however, very desirable, especially if observations in the southern hemisphere could be made use of. In my previous calculations I separated the summer from the winter months, and assuming what is known to be approximately the case, each hemisphere to behave alike when the sun occupies corresponding positions, I was enabled to form an expression of the potential applicable for the whole world simultaneously. But this is admittedly a defective process, and in drawing the equipotential curves I was careful for this reason to use only the averages taken over the whole year and to make no attempt to separate the two hemispheres. VON BEZOLD, who later, basing his calculations on my figures, effected this separation, is often quoted as having thus completed my own investigation, but his extension of my work, for the reason given, seems to me to be deceptive and to push too far the observed approximate symmetry in the two hemispheres. What I now regret, however, is that I did not divide the year into four parts instead of two, as Dr. CHREE'S results seem to show that the time of equinoxes deserves special consideration.

If the views here brought forward are correct, all peculiarities of the barometric variation should be reproduced in the magnetic effect, though we must remember that the converse does not hold, and that peculiarities of the magnetic effect depending partly on variations of electric conductivity need have no counterpart in the barometric changes. Thus the greater amplitude of the magnetic variation between summer and winter has already been ascribed to increased conductivity of the atmosphere during the summer. The close relationship between the two phenomena is confirmed by the increased amplitude observed in both near the time of equinoxes. The diurnal period of barometric pressure is known to have maxima at these epochs, and the valuable researches of Dr. CHREE have shown that these maxima are also found in the diurnal variation of the magnetic element. If we take the variation of declination as characteristic, Dr. CHREE'S formula for the semidiurnal term, leaving out the annual variation, is:

$$\delta D = 1 \cdot 82 \left[1 + 0 \cdot 137 \sin (2t + 271^\circ)\right],$$

where t is measured from the beginning of January and each month counts as 30 degrees. The corresponding term in the barometric formula is, according to HANN,*

$$\delta p = 0 \cdot 988 \left[1 + 0 \cdot 061 \sin (2t + 293 \cdot 4)\right],$$

but if I understand the formula correctly, the time here is counted from the middle of January. To make the equations correspond, we must therefore diminish the angle in the last equation by 30 degrees, reducing it to 263·4 degrees, in close agreement with that given by CHREE, the phase angle for the equinox being 265 degrees. The maxima of horizontal force occur, however, a fortnight later, so that too much value should not be given to this agreement; the effect in amplitude is only about half for the barometric variation; but questions of conductivity may affect this ratio.

A remarkable feature distinguishing the barometric change is the maximum which takes place simultaneously in both hemispheres early in January when the earth is in perihelion. According to the theory here discussed, a corresponding annual inequality should show itself in the magnetic variation, though the effect would be partially masked in the northern hemisphere by the changes of conductivity, and could only be ascertained by a comparison of the annual terms in the two hemispheres. We should expect the difference between winter and summer to be more marked in the southern hemisphere, because there the effects of conductivity would act in the same direction as the effects of diminished distance from the sun. It is much to be desired that some systematic attempt should be made to investigate the lunar influence on the magnetic changes, for we possess at present only the vaguest information as to how the different components of magnetic force are affected. It is quite possible that the effects may depend on a tidal disturbance of the upper regions of the atmosphere. If so, we may expect to get a valuable test of our theory by their investigation.

13. We are now prepared to discuss the magnitude of the conductivity required in order that the proposed theory should be tenable. If equations (9) and (10) are compared with each other, and the correction discussed in § 5 be applied, we find from the semi-diurnal term

$$\rho e = 3 \times 10^{-6}.$$

The first question which arises is the value to be assigned to e. Observations of the aurora borealis conducted by the Danish expedition under the late Mr. PAULSEN have led to the conclusion that the arc of these luminosities is generally at a height

* 'Met. Zeitschrift,' 1898, vol. XV., p. 381.

of 400 kilometres.* The height of meteors when they become luminous is as a rule over 100 kilometres, but there is one case on record in which a height of 780 kilometres was found. We may therefore take 300 kilometres as an outside limit for e, giving the value of 10^{-13} as the lower limit for the conductivity. This no doubt is a high value, and there may be some hesitation in accepting it as a possible one. Mr. C. T. R. WILSON has, however, already drawn attention to the fact that at high altitudes we must, with the same ionising power, expect a much increased conductivity, for the ionic velocity due to unit difference of potential varies inversely with the pressure. If, further, as the experiments indicate, the ionising power and rate of recombination of ions both diminish directly as the pressure, it would follow that when the pressure is only the millionth part of an atmosphere the conductivity should be for the same ionisation a million times as great as at the surface of the earth.

Researches on the conductivity of gases generally give relative measures, so that it is not always easy to infer its value in C.G.S. units, but I think the following examples will give an idea of the order of magnitude involved.

The quantity of electricity in the form of ions of each kind under normal conditions at the surface of the earth is $0 \cdot 33 \times 10^{-16}$ in electromagnetic measure. To obtain the conductivity the figures must be multiplied by the ionic velocity per unit fall of potential, the sum of the velocities of both kinds of ions being 3×10^{-8}. The conductivity of air at the surface of the earth is therefore under normal conditions 10^{-24}. GERDIEN, in one of his balloon ascents, determined the conductivity at a height of 6000 metres and found it to be 2×10^{-24}, which, as far as it goes, confirms the conclusion that the conductivity is inversely proportional to the density. At a height such that the pressure is one dyne per square centimetre, and assuming that the recombination of ions is not materially affected by the low temperature, we should thus get a conductivity of 10^{-18}, showing that, if the views discussed in this paper are correct, the ionising power at great altitudes must be considerably greater than that which acts on the air near the surface of the earth.

In speaking of the ionising effects of Röntgen rays, Professor J. J. THOMSON† states that even when the ionisation is exceptionally large the proportion of the number of free ions to the number of molecules of the gas is less than 1 to 10^{12}. From this I calculate the conductivity to be about 10^{-20} at atmospheric pressure. Some experiments by RUTHERFORD fix the conductivity of air, subject to the action of radium having an activity 1000 times less than pure radium, to be $0 \cdot 7 \times 10^{-19}$ under normal conditions. These figures would give to the conductivity of air at a pressure equal to that of a millionth atmosphere a magnitude comparable with that required. We know of much more powerful ionisers than the Röntgen tube or even radium. An electric discharge itself is sufficient to ionise a gas, as I proved as far back as

* 'Rapports du Congrès International de Physique,' vol. III., p. 438 (1900).

† 'Conductivity of Electricity through Gases,' p. 256.

1887. Data supplied by H. A. Wilson* show that in the positive column of a vacuum tube the conductivity reaches the value 10^{-13} and the kathode glow is even more highly conducting. The same author has experimented with air ionised in contact with hot platinum, and the data supplied by his diagrams† allow us to fix the conductivity of such air as about 4×10^{-17} at a temperature of 1080°. When the air was charged with the spray from a 1 per cent. solution of a potassium salt, the conductivity rose to $1 \cdot 4 \times 10^{-13}$, the temperature being 1200°. The conductivity of a Bunsen burner has been measured by Gold and found to be 8×10^{-15}. In view of these figures, which all apply of course to atmospheric pressure, we ought not, I think, to reject the value of 10^{-13} as an impossible one for the conductivity of air at high altitudes, but it is necessary to inquire into causes which produce so strong an ionisation.

The increased intensity of the magnetic variation during the summer months suggests directly that we are dealing with a solar action. This action may be simply an effect of radiation or it may be due to an injection of ions into the atmosphere. The former hypothesis is the one which presents itself as the most natural one, though the coronal streamers lend some countenance to the second view, which has often been put forward and sometimes even pressed in support of wildly speculative theories.

Ultra-violet radiation is known to ionise air in contact with metallic surfaces, but the evidence is somewhat conflicting as to the effect of radiation on the air itself. Unless the air is absolutely free of dust, the observed action may be due to the illumination of the dust and not of the air. Dust-free air is so transparent to luminous radiation that it would not be surprising if the ionising effect would disappear, as some experimenters believe it to do, when proper precautions are taken. On the other hand, Dr. V. Schumann has shown that air has a very strong absorbing power for wave-lengths which are sufficiently short. Such short wave-lengths are supplied by several metallic sparks, and are freely transmitted through hydrogen. Nevertheless it seems difficult to believe that, even if emitted by the hottest portion of the sun's envelope, they are not absorbed again by the surrounding cooler layers. We are not, therefore, at present in a position to assert that sufficiently short wave-lengths can enter the atmosphere and be absorbed in the outer layer, thereby causing ionisation, but we know so little about the conditions of the uppermost layers that we may reasonably retain the view that the powerful ionisation of the air, which we must consider to be an established fact, is a direct effect of solar radiation.

If we turn to the possibility of a direct injection of ions by the sun into our atmosphere, we have to deal with the alternatives of supposing that ions of both kinds are introduced or only those of one kind. The second alternative must be rejected at once, because a simple calculation shows that the outward force due to the volume

* 'Phil. Mag.,' 1900, p. 512.

† 'Phil. Trans.,' vol. 202, p. 243 (1904).

electrification of air necessary to account for the required conductivity would be more than sufficient to overcome gravitation and to drive out the conducting portions at an enormous rate. The injection, on the other hand, of a sufficient number of ions of both kinds also presents difficulties on account of the large quantity of new matter which would have to accumulate in our air, especially if it is considered that recombination at a rapid rate would take place both in the journey from the sun to the earth and in the passage through the different layers of the atmosphere. The only alternative to ultra-violet radiation seems, therefore, to lie in the injection of ions travelling with sufficient rapidity to generate other ions by impact. The air itself, according to this view, would supply the raw material for the ionisation, the injected corpuscles only acting as fertilisers. There are, of course, other possibilities, such as the introduction of radioactive matter, or a spontaneous ionisation which may, if the rate of recombination is slow, be very effective at a great height; but that the sun undoubtedly plays an important part in the process is shown not only by the summer effect, but also by the periodic changes of the magnetic variation, which corresponds with the sunspot cycle. I have held for many years and frequently expressed the opinion that the relationship can only be explained satisfactorily on the supposition that the electric conductivity at times of many sunspots is increased. Whether this is a direct influence of the sun, or only an indication that an ionising influence is brought into the solar system from outside at times of many sunspots, is a question which everyone is likely to answer according to his individual views of the cause of sunspot variability.

That the increase in the number of sunspots coincides with an increased conductivity of the upper layers of the atmosphere is also indicated by the eleven years' period of the aurora borealis. The distinguishing feature of the relationship seems to be this, that auroral displays extend further into moderate latitudes when the solar activity is great. An increase of conductivity is the simplest and most natural way of accounting for the effect. The primary cause of the electric discharges which manifest themselves in the aurora is still unknown. We may look for it, perhaps, in electrostatic forces which are always present, but causing a visible discharge only when their intensity rises abnormally, the course and intensity of discharge being much affected by inequalities of conducting power. On the other hand, there are other electromotive forces of induction not discussed in the paper, such as those accompanying a general drift of the atmosphere from west to east, which may well have something to do with the cause of auroral displays. Or again, if interplanetary space contains sufficient matter to be conducting, as I believe it must, there will be strong electromotive forces acting in the earth's magnetic field between the conducting powers rotating with the earth and those of interplanetary space.

Outbreaks of magnetic disturbances, affecting sometimes the whole of the earth simultaneously, may be explained by sudden local changes of conductivity which may extend through restricted or extensive portions of the atmosphere. I have shown in another place that the energy involved in a great magnetic storm is so considerable that we can only think of the earth's rotational energy as the source from which it ultimately is drawn. The earth can only act through its magnetisation in combination with the circulation of the atmosphere, so that magnetic storms may be considered to be only highly magnified and sudden changes in the intensity of electric currents circulating under the action of electric forces which are always present.

Those currents only are discussed in this communication which produce periodic variations in the magnetic elements, but there are also electromotive forces giving rise to current functions which are expressed by zonal harmonics and cannot under ordinary circumstances be observed, though any variation of conductivity between summer and winter would produce an annual period.

One further consequence of the theory deserves to be noticed. The electric currents indicated by our theory are sufficiently large to produce a sensible heating effect in the low-pressure regions through which they circulate. They will protect, therefore, the outer sheets of the atmosphere from falling to the extremely low temperatures which sometimes have been assumed to exist there, and they may help to form the isothermal layer which balloon observations have proved to exist at a height of about 50,000 feet.

Enough has been said to show the importance of the questions on which further investigation of the diurnal variation must give valuable information. If the fundamental ideas underlying the present enquiry stand the test of further research, we are in possession of a powerful method which will enable us to trace the cosmical causes which affect the ionisation of the upper regions of the atmosphere and which act apparently in sympathy with periodic effects showing themselves on and near the surface of the sun. It should be our endeavour to put the theory itself to a more accurate test than can at present be done. The most promising line of attack seems to me to be the investigation of the diurnal variation near the equator, where, as explained in § 6, it should not only vary with local time, but possess a term depending on the time of the meridian which passes through the magnetic axis. An exact determination of lunar effects would also, as has already been pointed out, serve as a valuable test of the theory.

PART II.

The problem to be solved may be stated thus: a spherical shell of fluid is animated by a quasi-tidal motion and is under the influence of magnetic forces of which only the vertical components are considered. It is required to find the magnetic effect of the induced currents if the motion is subject to a velocity potential $\psi_n^\sigma \cos \sigma (\lambda + t - \alpha)$, where ψ_n^σ is a surface harmonic, λ the longitude measured from some standard meridian towards the east, and t is the local time of that meridian. The conductivity ρ of the fluid is not necessarily uniform, but we take it to be expressible in the form

$$\rho = \rho_0 + \rho_1 \cos \theta + \rho_2 \sin \theta \cos (\lambda + t),$$

where θ is the colatitude, and $\lambda+t$ measures the difference in longitude between the sun and the place at which ρ is required. The question is solved if we can determine the current function of the electric currents which are generated by the fluid moving through the magnetic field. The problem for constant conductivity has been treated in the first part of this communication and the interest of a non-uniform conducting power is confined to the case that the variability depends on the angular distance between the sun and the point considered. If ω be this angle, the effect of the sun's radiation will be proportional to $\cos\omega$ in the hemisphere subject to the radiation, *i.e.* for values of ω smaller than $\frac{1}{2}\pi$. If the induction is due to the ionising power of the sun's rays, the rate of recombination of ions has to be considered, but unless this rate is of a different order of magnitude from that observed near the surface of the earth, the conductivity may be considered to be proportional everywhere to the illuminating power. For values of ω intermediate between $\frac{1}{2}\pi$ and π we must, then, give zero value to the conductivity. By means of Fourier series we may now express the conductivity in a series

$$\rho = \rho'_0\left[\frac{2}{\pi} + \tfrac{1}{2}\cos\omega + \frac{2}{3\pi}\cos 2\omega + \ldots\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

which satisfies the condition

$$\rho = \rho'_0 \cos\omega \text{ for } 0 < \omega < \frac{\pi}{2}, \quad \text{and} \quad \rho = 0 \text{ for } \frac{\pi}{2} < \omega < \pi.$$

Confining ourselves to the first two terms and substituting the value of $\cos\omega$ from (13) in terms of the hour-angle of the sun and its declination, we obtain

$$\rho = \rho'_0\left[\frac{2}{\pi} + \tfrac{1}{2}\sin\delta\cos\theta + \tfrac{1}{2}\cos\delta\sin\theta\cos(\lambda+t)\right].$$

The conductivity has therefore the assumed form if we put

$$\rho_0 = \frac{2}{\pi}\rho'_0; \quad \rho_1 = \tfrac{1}{2}\rho'_0\sin\delta; \quad \rho_2 = \tfrac{1}{2}\rho'_0\cos\delta.$$

Were we to adopt the simpler form and put the conductivity proportional to $1+\cos\omega$, so that it reaches zero value only at midnight, we should have to put

$$\rho_1 = \rho_0\sin\delta; \quad \rho_2 = \rho_0\cos\delta,$$

and in every case ρ can be expressed in terms of a series such as (17), our investigation by proper adjustment of the constants taking account of the first two terms. The term in $\cos 2\omega$ might be taken into consideration without much difficulty should that become necessary. The value of ρ_0 can provisionally be put equal to unity and re-introduced at a later stage. Writing $\gamma' = \rho_1/\rho_0$ and $\gamma = \rho_2/\rho_0$ we may therefore put

$$\rho = 1 + \gamma'\cos\theta + \gamma\sin\theta\cos(\lambda+t) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18).$$

In order to avoid frequent interruption, I prove in the first instance a few formulæ of transformation which I have found of great utility in these investigations. I start from the following equations denoted in my previous communications by Roman letters, which it is convenient to retain :

$$(2n+1)\cos\theta Q_n^\sigma = (n-\sigma+1)\,Q^\sigma_{n+1}+(n+\sigma)\,Q^\sigma_{n-1} \quad . . . . (A),$$

$$(2n+1)\sin\theta Q_n^\sigma = Q_{n+1}^{\sigma+1}-Q_{n-1}^{\sigma+1} \quad . . . . (B),$$

$$= (n+\sigma)(n+\sigma-1)\,Q_{n-1}^{\sigma-1}-(n-\sigma+2)(n-\sigma+1)\,Q_{n+1}^{\sigma-1} \quad . (C),$$

$$\frac{2\sigma Q_n^\sigma}{\sin\theta} = (n+\sigma)(n+\sigma-1)\,Q_{n-1}^{\sigma-1}+Q_{n-1}^{\sigma+1} \quad . . . . (D),$$

$$= Q_{n+1}^{\sigma+1}+(n-\sigma+2)(n-\sigma+1)\,Q_{n+1}^{\sigma-1} \quad . . . . (E),$$

$$\frac{2dQ_n^\sigma}{d\theta} = (n+\sigma)(n-\sigma+1)\,Q_n^{\sigma-1}-Q_n^{\sigma+1} \quad . . . . (H_1).$$

Q_n^σ denotes the tesseral function derived from the zonal harmonic P_n by the relation $Q_n^\sigma = \sin^\sigma\theta\,\frac{d^\sigma P_n}{d\mu^\sigma}$, where $\mu = \cos\theta$.

Multiplying (D) by $(n-\sigma+1)$ and (E) by $(n+\sigma)$, and adding, we find, with the help of (A),

$$(2n+1)\left[(n+\sigma)(n-\sigma+1)\cos\theta Q_n^{\sigma-1}-\frac{2\sigma}{\sin\theta}\,Q_n^\sigma\right] = -(n+\sigma)\,Q_{n+1}^{\sigma+1}-(n-\sigma+1)\,Q_{n-1}^{\sigma+1}.$$

If in the formula (A) we substitute $\sigma+1$ for σ, it becomes

$$(2n+1)\cos\theta Q_n^{\sigma+1} = (n-\sigma)\,Q_{n+1}^{\sigma+1}+(n+\sigma+1)\,Q_{n-1}^{\sigma+1}.$$

From the last two equations we obtain, by subtraction and substitution of (H_1),

$$(2n+1)\cos\theta\,\frac{dQ_n^\sigma}{d\theta}-\frac{\sigma}{\sin\theta}\,Q_n^\sigma = -nQ_{n+1}^{\sigma+1}-(n+1)\,Q_{n-1}^{\sigma+1}.$$

Now multiplying (B) by $n\,.\,(n+1)$, and subtracting, we finally obtain

$$\sin\theta\cos\theta\frac{dQ_n^\sigma}{d\theta}-n\,.\,n+1\,.\sin^2\theta Q_n^\sigma-\sigma Q_n^\sigma$$

$$= \sin\theta\,\frac{(n-1)(n+1)\,Q_{n-1}^{\sigma+1}-n\,(n+2)Q_{n+1}^{\sigma+1}}{2n+1} \quad . . . (K_1).$$

If on the right-hand side of (K_1) we substitute the values of $Q_{n+1}^{\sigma+1}$ and $Q_{n-1}^{\sigma+1}$ from D and E, we obtain a corresponding equation

$$\sin\theta\cos\theta\,\frac{dQ_n^\sigma}{d\theta}-n\,.\,n+1\,.\sin^2\theta\,.\,Q_n^\sigma+\sigma Q_n^\sigma$$

$$= \frac{\sin\theta}{2n+1}\left\{n(n+2)(n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1}-(n-1)(n+1)(n+\sigma)(n+\sigma-1)\,Q_{n-1}^{\sigma-1}\right\}$$

$$. . . (K_2).$$

A further useful transformation is derived from the equations

$$(2n+1)\,\sigma\cos\theta Q_n^{\sigma} = \sigma\,(n-\sigma+1)\,Q^{\sigma}_{n+1}+\sigma\,(n+\sigma)\,Q^{\sigma}_{n-1},$$

$$(2n+1)\sin\theta\,\frac{dQ_n^{\sigma}}{d\theta} = n\,.\,(n-\sigma+1)\,Q^{\sigma}_{n+1}-(n+1)\,(n+\sigma)\,Q^{\sigma}_{n-1}.$$

If we subtract and add these equations, they reduce to

$$\sigma\cos\theta Q_n^{\sigma}-\sin\theta\frac{dQ_n^{\sigma}}{d\theta} = \sin\theta Q_n^{\sigma+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (L_1),$$

and

$$\sigma\cos\theta Q_n^{\sigma}+\sin\theta\frac{dQ_n^{\sigma}}{d\theta} = (n+\sigma)\,(n-\sigma+1)\sin\theta Q_n^{\sigma-1} \quad . \quad . \quad . \quad . \quad (L_2).$$

We shall require to find the effect of the operation

$$\left(\frac{d}{d\lambda}\cos\lambda\frac{d}{d\lambda}+\cos\lambda\frac{d}{d\theta}\sin^2\theta\frac{d}{d\theta}\right)Q_n^{\sigma}\cos(\sigma\lambda-\alpha) \quad . \quad . \quad . \quad . \quad (19).$$

We omit, for the sake of shortness, temporarily the constant α, and divide the operation into two parts, the first being

$$\cos\lambda\left(\frac{d^2}{d\lambda^2}+\sin\theta\frac{d}{d\theta}\sin\theta\frac{d}{d\theta}\right)Q_n^{\sigma}\cos\sigma\lambda.$$

From the fundamental equation relating to tesseral harmonics this is equal to

$$-\tfrac{1}{2}n\,.\,(n+1)\sin^2\theta Q_n^{\sigma}\left[\cos(\sigma+1)\lambda+\cos(\sigma-1)\lambda\right] \quad . \quad . \quad . \quad . \quad (20).$$

The remaining part of the operation is

$$\sigma\sin\lambda\sin\sigma\lambda Q_n^{\sigma}+\sin\theta\cos\theta\cos\lambda\cos\sigma\lambda\,\frac{d}{d\theta}\,Q_n^{\sigma}$$

$$=\tfrac{1}{2}\cos(\sigma+1)\lambda\,(\sin\theta\cos\theta\,dQ_n^{\sigma}-\sigma Q_n^{\sigma})+\tfrac{1}{2}\cos(\sigma-1)\lambda\left(\sin\theta\cos\theta\frac{dQ_n^{\sigma}}{d\theta}+\sigma Q_n^{\sigma}\right) \;.\; (21).$$

If (20) and (21) are now added, and K_1 and K_2 are applied, we find the result of the operation to be, restoring α,

$$\frac{\sin\theta}{2\,.\,2n+1}\left\{(n-1)\,(n+1)\,Q_{n-1}^{\sigma+1}-n\,.\,n+2\,.\,Q_{n+1}^{\sigma+1}\right\}\cos\left\{(\sigma+1)\lambda-\alpha\right\}$$

$$+\frac{\sin\theta}{2\,.\,2n+1}\left\{n(n+2)(n-\sigma+2)(n-\sigma+1)Q_{n+1}^{\sigma-1}-(n-1)(n+1)(n+\sigma)(n+\sigma-1)Q_{n-1}^{\sigma-1}\right\}$$

$$\cos\left\{(\sigma-1)\lambda-\alpha\right\} \quad . \quad . \quad . \quad (22).$$

We may note that each of the equations used, and therefore the final results, remains true for $\sigma = 0$, if we define

$$n\,.\,n+1\,.\,Q_n^{-1} = -Q'_n.$$

This is in agreement with RODRIGUEZ's theorem, if the definition of Q_n^σ depending on the operation

$$(1-\mu^2)^{\frac{\sigma}{2}} d^{n+\sigma} (\mu^2-1)^n / 2^n n! \, d\mu^{n+\sigma}$$

is extended to negative values of σ, for in that case

$$\begin{aligned} Q_n^{-\sigma} &= \frac{1}{2^n n!} (1-\mu^2)^{-\frac{\sigma}{2}} \frac{d^{(n-\sigma)} (\mu^2-1)^n}{d\mu^{n-\sigma}} \\ &= (-1)^\sigma \frac{1}{2^n n!} \frac{(n-\sigma)!}{(n+\sigma)!} \frac{d^{n+\sigma} (\mu^2-1)^n}{d\mu^{n+\sigma}} \\ &= (-1)^\sigma \frac{(n-\sigma)!}{(n+\sigma)!} Q_n^\sigma \quad \ldots \ldots \ldots \ldots \quad (23). \end{aligned}$$

It follows that the operation (17), in the case where $Q_n^0 \cos \alpha$ replaces $Q_n^\sigma \cos (\sigma\lambda - \alpha)$, reduces to

$$\frac{\sin \theta}{2 \, . \, 2n+1} \{(n-1)(n+1) Q'_{n-1} - n(n+2) Q'_{n+1}\} [\cos (\lambda-\alpha) + \cos (\lambda+\alpha)].$$

This result may also easily be obtained independently, but in view of the ultimate application of (22) it is important to include the special case in the general one.

It will be appropriate here to obtain another formula which will be used subsequently. Let it be required to find

$$\cot \theta \frac{d^2 Q_n^\sigma}{d\lambda^2} + \frac{d}{d\theta} \sin \theta \cos \theta \frac{dQ_n^\sigma}{d\theta} \quad \ldots \ldots \ldots \quad (24).$$

From the fundamental equation we find this to be equal to

$$-n \, . \, n+1 \, . \sin \theta \cos \theta Q_n^\sigma - \sin^2 \theta \frac{dQ_n^\sigma}{d\theta},$$

and as

$$-n \, . \, n+1 \, . \cos \theta Q_n^\sigma = -\frac{n \, . \, n+1}{2n+1} \{(n+\sigma) Q^\sigma_{n+1} + (n-\sigma+1) Q^\sigma_{n+1}\}$$

and

$$-\sin \theta \frac{dQ_n^\sigma}{d\theta} = \frac{1}{2n+1} \{(n+1)(n+\sigma) Q^\sigma_{n-1} - n(n-\sigma+1) Q^\sigma_{n+1}\},$$

(24) becomes equal to

$$-\frac{\sin \theta}{2n+1} \{(n+2) n (n-\sigma+1) Q^\sigma_{n+1} + (n-1)(n+1)(n+\sigma) Q^\sigma_{n-1}\} \quad . \quad . \quad (25).$$

We are now in a position to attack our main problem. The equations to be solved are

$$\rho X = \rho \frac{dS}{d\theta} + \frac{dR}{\sin \theta \, d\lambda}, \quad \rho Y = \rho \frac{dS}{\sin \theta \, d\lambda} - \frac{dR}{d\theta} \quad . \quad . \quad . \quad . \quad . \quad (14),$$

where X and Y have the values given in (1), and for ρ we must substitute the expression (18), remembering, however, that we must ultimately restore the factor ρ_0. We may also temporarily omit the factor C in the expression for the electric forces. To find the current function R we must eliminate S, but owing to the fact that ρ contains θ and λ, this does not seem to be possible directly. The difficulty must be turned by eliminating in the first place R, and if S is then found, R may be determined from the first of the above equations. It may occur to the reader that R might be more directly obtained if the resistivity were introduced instead of the conductivity. This is true, but the results are less valuable, as may be seen from the fact that, as suggested above, the Fourier expansion may have to be applied to the conducting power in so far as it depends on the position of the sun. If the resistivity were introduced as the variable, the high and possibly infinite values which the factor would take when the conductivity sinks low or vanishes would present difficulties much greater than those met with by keeping the conductivity as the variable quantity.

The elimination of R requires in the first instance the reduction of

$$\frac{d}{d\theta}\rho \mathrm{X} \sin\theta + \frac{d}{d\lambda}\rho \mathrm{Y},$$

i.e. of

$$\rho\left(\frac{d}{d\theta}\mathrm{X}\sin\theta + \frac{d}{d\lambda}\mathrm{Y}\right) + \mathrm{X}\sin\theta\frac{d\rho}{d\theta} + \mathrm{Y}\frac{d\rho}{d\lambda}.$$

Introducing the values of X and Y, the operation reduces to

$$-(1+\gamma'\cos\theta+\gamma\sin\theta\cos\lambda)\sin\theta\frac{d\psi}{d\lambda} - \gamma'\sin\theta\cos\theta\frac{d\psi}{d\lambda}$$
$$-\gamma\cos\theta\left(\cos\theta\cos\lambda\frac{d\psi}{d\lambda} - \sin\theta\sin\lambda\frac{d\psi}{d\theta}\right).$$

The part independent of γ and γ' is

$$-\sin\theta\frac{d\psi}{d\lambda} = -\sigma\sin\theta\psi_n^\sigma\cos(\sigma\lambda-\alpha).$$

The part depending on γ' is

$$-2\gamma'\sin\theta\cos\theta\frac{d\psi}{d\lambda} = -2\gamma'\sigma\sin\theta\cos\theta\psi_n^\sigma\cos(\sigma\lambda-\alpha)$$
$$= -2\gamma'\sigma\sin\theta\frac{(n-\sigma+1)\,\psi^\sigma{}_{n+1} + (n+\sigma)\,\psi^\sigma{}_{n-1}}{2n+1}.$$

We are left with the part dependent on γ, the factor of which is

$$\cos\lambda\frac{d\psi}{d\lambda}(\cos^2\theta - \sin^2\theta) + \sin\theta\cos\theta\sin\lambda\frac{d\psi}{d\theta}.$$

Of this we take separately

$$-\sin^2\theta\cos\lambda\frac{d\psi}{d\lambda}$$

$$= -\frac{\sin^2\theta}{2}[\cos\{(\sigma+1)\lambda-\alpha\}+\cos\{(\sigma-1)\lambda-\alpha\}]\psi_n{}^\sigma$$

$$= -\sigma\sin\theta\frac{\psi_{n+1}{}^{\sigma+1}-\psi_{n-1}{}^{\sigma+1}}{2\,.\,2n+1}\cos\{(\sigma+1)\lambda-\alpha\}$$

$$= -\sigma\sin\theta\frac{(n+\sigma)(n+\sigma-1)\psi_{n-1}{}^{\sigma-1}-(n-\sigma+2)(n-\sigma+1)\psi_{n+1}{}^{\sigma-1}}{2\,.\,2n+1}\cos\{(\sigma-1)\lambda-\alpha\}.$$

The remaining terms depending on γ are

$$\tfrac{1}{2}\frac{\cos\theta}{2n+1}\Big[\sigma\cos\theta\psi_n{}^\sigma(\cos\{(\sigma+1)\lambda-\alpha\}+\cos\{(\sigma-1)\lambda-\alpha\})$$
$$+\sin\theta\frac{d\psi_n{}^\sigma}{d\theta}(\cos\{(\sigma-1)\lambda-\alpha\}-\cos\{(\sigma+1)\lambda-\alpha\})\Big],$$

or, making use of (L_1) and (L_2),

$$\tfrac{1}{2}\frac{\cos\theta\sin\theta}{2n+1}\,\psi_n{}^{\sigma+1}\cos[(\sigma+1)\lambda-\alpha]+(n+\sigma)(n-\sigma+1)\,\psi_n{}^{\sigma-1}\cos[(\sigma-1)\lambda-\alpha].$$

The terms containing γ, leaving out the longitude factors, are therefore

$$\frac{\gamma\sin\theta}{2\,.\,2n+1}[(n-2\sigma)\,\psi_{n+1}{}^{\sigma+1}+(n+2\sigma+1)\,\psi_{n-1}{}^{\sigma+1}+(n-\sigma+1)(n-\sigma+2)(n+2\sigma)\,\psi_{n+1}{}^{\sigma-1}$$
$$+(n+\sigma)(n+\sigma-1)(n-2\sigma+1)\,\psi_{n-1}{}^{\sigma-1}].$$

Collecting our results together, we find as the effect of the operation, eliminating R on the left-hand side of equations (14),

$$\gamma\sin\theta\frac{(n-\sigma+1)(n-\sigma+2)(n+2\sigma)\,\psi_{n+1}{}^{\sigma-1}+(n+\sigma)(n+\sigma-1)(n-2\sigma+1)\,\psi_{n-1}{}^{\sigma-1}}{2\,.\,2n+1}\cos\{(\sigma-1)\lambda-\alpha\}$$

$$-\sigma\sin\theta\left[\psi_n{}^\sigma+2\gamma'\frac{(n-\sigma+1)\,\psi^\sigma{}_{n+1}+(n+\sigma)\,\psi^\sigma{}_{n-1}}{2n+1}\right]\cos(\sigma\lambda-\alpha)$$

$$+\gamma\sin\theta\frac{(n-2\sigma)\,\psi_{n+1}{}^{\sigma+1}+(n+2\sigma+1)\,\psi_{n-1}{}^{\sigma+1}}{2\,.\,2n+1}\cos\{(\sigma+1)\lambda-\alpha\}\quad . \quad . \quad . \quad . \quad . \quad (26).$$

The expression reduces to about half its terms when $n=\sigma$, and for the two special cases which form the main subject of the present inquiry we have then

Case I. $n=1$, $\sigma=1$.

$$\frac{1}{\sin\theta}\left(\frac{dP\rho\sin\theta}{d\theta}+\frac{dQ\rho}{d\lambda}\right)=\gamma\psi_2{}^0\cos\alpha-(\psi_1{}^1+\tfrac{2}{3}\gamma'\psi_2{}^1)\cos(\lambda-\alpha)-\tfrac{1}{6}\gamma\psi_2{}^2\cos(2\lambda-\alpha).$$

Case II. $n = 2$, $\sigma = 2$.

$$\frac{1}{\sin\theta}\frac{dP\rho\sin\theta}{d\theta} + \frac{dQ\rho}{d\lambda} = \tfrac{6}{5}\gamma\left(-\psi_1^1+\psi_3^1\right)\cos(\lambda-\alpha) - \left(2\psi_2^2+\tfrac{4}{5}\gamma'\psi_3^2\right)\cos(2\lambda-\alpha) - \tfrac{1}{5}\gamma\psi_3^3\cos(3\lambda-\alpha).$$

Our next step must be to find the expression resulting from the terms containing S in the elimination of R.

We shall begin by assuming S to be a spherical harmonic of the form Q_n^σ, and shall again take the parts depending on γ, γ' separately. Independently of both these quantities, we have

$$\frac{d}{d\theta}\sin\theta\frac{dQ_n^\sigma}{d\theta} + \frac{1}{\sin\theta}\frac{d^2Q_n^\sigma}{d\lambda^2} = -n\,.\,\overline{n+1}\,.\,\sin\theta Q_n^\sigma \quad . \quad . \quad . \quad . \quad (27).$$

As factor of γ' we have

$$\frac{d}{d\theta}\sin\theta\cos\theta\frac{dQ_n^\sigma}{d\theta} + \cot\theta\frac{d^2Q_n^\sigma}{d\lambda^2}.$$

The value of this has been obtained under (25).

Finally, as factor of γ, we have

$$\frac{d}{d\theta}\sin^2\theta\cos\lambda\frac{dQ_n^\sigma}{d\theta} + \frac{d}{d\lambda}\cos\lambda\frac{dQ_n^\sigma}{d\lambda}.$$

This is identical with the expression (19), the result of the operation being given by (22).

If we collect our results, by adding the right-hand side of (27) to (25) and (22) after applying the appropriate factor, we shall have obtained an expression for

$$\frac{d}{d\theta}\rho\sin\theta\frac{dQ_n^\sigma}{d\theta} + \frac{1}{\sin\theta}\cdot\frac{d}{d\lambda}\cdot\rho\frac{dQ_n^\sigma}{d\lambda} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (28).$$

It will appear that S can be expressed in the form of a series

$$S = \kappa_n^0\cos\alpha Q_n^0 + \sum_{\sigma=1}^{\sigma=\infty}\left\{\kappa_n^\sigma\cos(\sigma\lambda-\alpha) + \mu_n^\sigma\cos(\sigma\lambda+\alpha)\right\}Q_n^\sigma \quad . \quad . \quad . \quad (29),$$

where α is determined by the phase of the velocity potential $\psi_n^\sigma\sin(\sigma\lambda-\sigma)$, which rules the flow of matter in which the electric currents are induced. We shall avoid the labour involved in the consideration of special cases if we write (29) in the form

$$S = \sum_{\sigma=-\infty}^{\sigma=+\infty}\kappa_n^\sigma Q_n^\sigma\cos(\sigma\lambda-\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30).$$

Adopting the definition (23) for Q_n^σ, where σ is negative, we may return to the original form by replacing the μ coefficients with the help of

$$\kappa_n^{-\sigma} = (-1)^\sigma \frac{(n+\sigma)!}{(n-\sigma)!}\mu_n^\sigma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31).$$

If each term of (30) be subjected separately to the operation (28) and the results collected, so that all terms depending on any one value Q_n^σ are brought together, we may express the result of the operation by a series of the form

$$-\sum_{\sigma=-\infty}^{\sigma=+\infty} E_n^\sigma Q_n^\sigma \sin\theta \cos(\sigma\lambda-\alpha) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32),$$

which must be equal to (26).

If we put

$$E_n^\sigma = A_n^\sigma + B_n^\sigma\gamma' + C_n^\sigma\gamma \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (33),$$

we find

$$A_n^\sigma = n(n+1)\kappa_n^\sigma \quad . \quad (34),$$

$$B_n^\sigma = \frac{(n-1)(n+1)(n-\sigma)}{2n-1}\kappa^\sigma_{n-1} + \frac{n(n+2)(n+\sigma+1)}{2n+3}\kappa^\sigma_{n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35),$$

$$C_n^\sigma = \frac{(n-1)(n+1)}{2\,.\,2n-1}\kappa_{n-1}^{\sigma-1} - \frac{n\,.\,n+2}{2\,.\,2n+3}\kappa_{n+1}^{\sigma-1} + \frac{n\,.\,(n+2)(n+\sigma+1)(n+\sigma+2)}{2\,.\,2n+3}\kappa_{n+1}^{\sigma+1}$$

$$-\frac{(n-1)(n+1)(n-\sigma)(n-\sigma-1)}{2\,.\,2n-1}\kappa_{n-1}^{\sigma+1} \quad . \quad . \quad . \quad (36).$$

The values of κ which determine S may now be found by equating the factors of $\sin\theta Q_n^\sigma$ in (33) to the corresponding factor of $\sin\theta\psi_n^\sigma$ in (26), remembering, however, that in the latter equation σ and n have the definite value belonging to the assumed velocity potential. If, for the sake of clearness, the type and degree of this velocity potential be now denoted by τ and m, we find

$$\left.\begin{array}{ll} E_m^\tau = \tau & 2(2m+1)E_{m-1}^{\tau-1} = -(m+\tau)(m+\tau-1)(m-2\tau+1)\gamma \\ (2m+1)E^\tau_{m-1} = 2\tau(m+\tau)\gamma' & 2(2m+1)E_{m+1}^{\tau-1} = -(m+2\tau)(m-\tau+1)(m-\tau+2)\gamma \\ (2m+1)E^\tau_{m+1} = 2\tau(m-\tau+1)\gamma' & 2(2m+1)E_{m-1}^{\tau+1} = -(m+2\tau+1)\gamma \\ & 2(2m+1)E_{m+1}^{\tau+1} = -(m-2\tau)\gamma \end{array}\right\} (37).$$

If in any of these values of E the index is less than the suffix, that value may be put equal to zero, as the tesseral function to which that coefficient applies is zero. All other values of E_n^σ are zero. If we take the case $\tau = 1$, $m = 1$, as an example, we obtain from (33), (34), (35), and (36) a number of equations in which all values of E_n^σ may be put equal to zero excepting E_1^1, E_2^1, E_2^0, E_2^2; the values of these are obtained from (37) and we thus find equation (33) for these special values of n and σ to become

$$2\kappa_1^{\ 1}+\tfrac{9}{5}\gamma'\kappa_2^{\ 1}-\gamma\left(\tfrac{3}{10}\kappa_2^{\ 0}-\tfrac{18}{5}\kappa_2^{\ 2}\right) = 1,$$
$$6\kappa_2^{\ 1}+\gamma'\left(\kappa_1^{\ 1}+\tfrac{32}{7}\kappa_3^{\ 1}\right)+\gamma\left(\tfrac{1}{2}\kappa_1^{\ 0}-\tfrac{4}{7}\kappa_3^{\ 0}+\tfrac{80}{7}\kappa_3^{\ 2}\right) = \tfrac{2}{3}\gamma',$$
$$6\kappa_2^{\ 0}+\gamma'\left(2\kappa_1^{\ 0}+\tfrac{24}{7}\kappa_3^{\ 0}\right)+\gamma\left(\tfrac{48}{7}\kappa_3^{\ 1}-\kappa_1^{\ 1}\right) = -\gamma,$$
$$6\kappa_2^{\ 2}+\tfrac{40}{7}\gamma'\kappa_3^{\ 2}+\gamma\left(\tfrac{1}{2}\kappa_1^{\ 1}-\tfrac{4}{7}\kappa_3^{\ 1}+\tfrac{120}{7}\kappa_3^{\ 3}\right) = \tfrac{1}{6}\gamma.$$

These and all similar equations in which the right-hand side is equated to zero are sufficient to determine the κ coefficients, each in terms of a series proceeding by powers of γ and γ'. I proceed to show how the successive approximations may be obtained. If γ and γ' are both zero, the first of the above equations leads to $\kappa_1^{\ 1} = \frac{1}{2}$, and, as the equation must hold for all values of γ, this gives us that portion of $\kappa_1^{\ 1}$ which is independent of γ and γ'. The remaining equations tell us that there can be no other factor κ which has a term not containing γ and γ'. The last three equations contain $\kappa_1^{\ 1}$ and they are the only equations out of the complete series which contain this particular factor. Substituting its value as far as it has been found and neglecting in the brackets all factors except $\kappa_1^{\ 1}$, because they must all depend on γ or γ', and therefore introduce quantities of the second order, we obtain a set of three equations which determines those coefficients which involve the first powers of these quantities. We are thus led to

$$\kappa_2^{\ 0} = -\tfrac{1}{12}\gamma\,; \quad \kappa_2^{\ 1} = \tfrac{1}{36}\gamma'\,; \quad \kappa_2^{\ 2} = \tfrac{1}{72}\gamma.$$

No other coefficients can contain terms depending on first powers. If we now write down all equations in which $\kappa_2^{\ 0}$, $\kappa_2^{\ 1}$, $\kappa_2^{\ 2}$ occur, we may determine the terms involving γ^2, $\gamma\gamma'$ and γ'^2. Thus if the above equation involving $\kappa_1^{\ 1}$ is reconsidered, we find that in view of our knowledge just acquired $\kappa_1^{\ 1}$ must contain terms in γ^2 and γ'^2 satisfying the equation

$$2\kappa_1^{\ 1}+\tfrac{1}{20}\gamma'^2-\tfrac{1}{40}\gamma^2 = 0.$$

The equations for E'_3, $E_3^{\ 3}$, $E_3^{\ -1}$, give

$$12\kappa_3^{\ 1}+\gamma'\left(\tfrac{16}{5}\kappa_2^{\ 1}+\tfrac{25}{3}\kappa_4^{\ 1}\right)+\gamma\left(\tfrac{4}{5}\kappa_2^{\ 0}-\tfrac{5}{6}\kappa_4^{\ 0}+25\kappa_4^{\ 2}-\tfrac{8}{5}\kappa_2^{\ 2}\right) = 0,$$
$$12\kappa_3^{\ 2}+\gamma'\left(\tfrac{8}{5}\kappa_2^{\ 2}+10\kappa_4^{\ 2}\right)+\gamma\left(\tfrac{4}{5}\kappa_2^{\ 1}-\tfrac{5}{6}\kappa_4^{\ 1}+35\kappa_4^{\ 3}\right) = 0,$$
$$12\kappa_3^{\ 3}+\tfrac{35}{3}\gamma'\kappa_4^{\ 3}+\gamma\left(\tfrac{4}{5}\kappa_2^{\ 2}-\tfrac{5}{6}\kappa_4^{\ 2}+\tfrac{140}{3}\kappa_4^{\ 4}\right) = 0,$$
$$12\kappa_3^{\ -1}+\gamma'\left(\tfrac{32}{5}\kappa_2^{\ -1}+5\kappa_4^{\ -1}\right)+\gamma\left(\tfrac{4}{5}\kappa_2^{\ -2}-\tfrac{5}{6}\kappa_4^{\ -2}+10\kappa_4^{\ 0}-\tfrac{48}{5}\kappa_2^{\ 0}\right) = 0.$$

The factors such as $\kappa_4^{\ 0}$, $\kappa_2^{\ -2}$, $\kappa_4^{\ -2}$, which can only contain powers of γ and γ' higher than the first may be left out of account in solving these equations, and we thus find all terms which contain γ^2, γ'^2 or $\gamma\gamma'$. We may proceed in this manner, gradually working by successive approximations from lower to higher powers. The following two tables contain the results, including all powers as far as the third, for the two typical atmospheric motions represented by the current functions $\psi_1^{\ 1}$ and $\psi_2^{\ 2}$. For convenience of use the μ coefficients now replace the κ coefficients with negative indices.

Values of κ_n^σ. (Velocity Potential $= \psi_1^1$.)

$n =$	1	2	3	4
$\sigma = 0$	$\frac{1}{40}\gamma\gamma'$	$-\frac{1}{12}\gamma - \frac{89}{3024}\gamma\gamma'^2 - \frac{2}{189}\gamma^3$	$\frac{2}{45}\gamma\gamma'$	$\frac{1}{168}\gamma^3 - \frac{1}{42}\gamma\gamma'^2$
1	$\frac{1}{2} + \frac{1}{80}\gamma^2 - \frac{1}{40}\gamma'^2$	$\frac{1}{36}\gamma' + \frac{89}{48 \times 189}\gamma'^3 - \frac{25}{48 \times 189}\gamma'\gamma^2$	$\frac{1}{270}\gamma^2 - \frac{1}{135}\gamma'^2$	$-\frac{1}{280}\gamma'\gamma^2 + \frac{1}{420}\gamma'^3$
2	—	$-\frac{1}{72}\gamma - \frac{121}{189 \times 192}\gamma^3 - \frac{25}{96 \times 189}\gamma\gamma'^2$	0	$-\frac{1}{14 \times 720}\gamma^3 + \frac{1}{14 \times 180}\gamma\gamma'^2$
3	—	—	$\frac{1}{1080}\gamma^2$	$-\frac{1}{14 \times 720}\gamma'\gamma^2$
4	—	—	—	$-\frac{1}{72 \times 280}\gamma^3$

Values of μ_n^σ. (Velocity Potential $= \psi_1^1$.)

$n =$	1	2	3	4
$\sigma = 1$	$-\frac{\gamma^2}{80}$	0	$\frac{\gamma^2}{180}$	$-\frac{1}{240}\gamma'\gamma^2$
$\sigma = 2$	—	$\frac{19}{64 \times 189}\gamma^3$	0	$\frac{1}{14 \times 240}\gamma^3$

Values of κ_n^σ. (Velocity Potential $= \psi_2^2$.)

$n =$	1	2	3	4	5
$\sigma = 0$	$-\frac{4}{105}\gamma^2\gamma'$	$\frac{4}{63}\gamma^2$	$-\frac{29}{720}\gamma^2\gamma'$	$-\frac{1}{28}\gamma^2$	$\frac{4}{105}\gamma'\gamma^2$
1	$-\frac{1}{105}\gamma^3$	0	$-\frac{1}{18}\gamma - \frac{133}{36 \times 360}\gamma^3 - \frac{133}{36 \times 360}\gamma\gamma'^2$	$\frac{1}{40}\gamma\gamma'$	$\frac{1}{350}\gamma^3 - \frac{2}{175}\gamma\gamma'^2$
2	—	$\frac{1}{3} + \frac{2}{189}\gamma^2 - \frac{4}{189}\gamma'^2$	$\frac{1}{45}\gamma' + \frac{11}{1620}\gamma'^3 + \frac{1}{48 \times 270}\gamma'\gamma^2$	$\frac{1}{420}\gamma^2 - \frac{1}{210}\gamma'^2$	$-\frac{1}{525}\gamma'\gamma^2 + \frac{2}{1575}\gamma'^3$
3	—	—	$-\frac{1}{180}\gamma - \frac{11}{36 \times 180}\gamma^3 + \frac{43}{16 \times 36 \times 45}\gamma\gamma'^2$	$-\frac{1}{1680}\gamma\gamma'$	$-\frac{1}{12600}\gamma^3 + \frac{1}{315}\gamma\gamma'^2$
4	—	—	—	$\frac{1}{3360}\gamma^2$	0
5	—	—	—	—	$-\frac{1}{45 \times 1680}\gamma^3$

VALUES of μ_n^σ. (Velocity Potential $= \psi_2^2$.)

$n =$	1	3	5
$\sigma = 1$	$\frac{1}{105}\gamma^3$	$-\frac{29}{12 \times 360}\gamma^3$	$\frac{1}{630}\gamma^3$

The determination of S is only of interest as a stepping stone to the evaluation of R. We must therefore return to the first of equations (14), and by its means determine $d\text{R}/d\lambda$ as a series of harmonics in the normal form.

If we write the velocity potential $\psi_m^\tau \sin(\tau\lambda - \alpha)$, we find by means of the formulæ of transformation previously introduced

$$\rho \cos\theta \frac{d\psi}{d\lambda} = \frac{(m-\tau+1)\,\psi^\tau_{m+1} + (m+\tau)\,\psi^\tau_{m-1}}{2m+1}\,\tau\cos(\tau\lambda-\alpha)$$

$$+\left\{(m-\tau+1)\frac{(m-\tau+2)\,\psi^\tau_{m+2}+(m+\tau+1)\,\psi_m^\tau}{(2m+1)(2m+3)}\right.$$

$$\left.+(m+\tau)\frac{(m-\tau)\,\psi_m^\tau+(m+\tau-1)\,\psi^\tau_{m-2}}{(2m-1)(2m+1)}\right\}\gamma'\tau\cos(\tau\lambda-\alpha)$$

$$+\left\{(m-\tau+1)\frac{(\psi_{m+2}{}^{\tau+1}-\psi_m{}^{\tau+1})}{2\,(2m+1)(2m+3)}\right.$$

$$\left.+(m+\tau)\frac{(\psi_m{}^{\tau+1}-\psi_{m-2}{}^{\tau+1})}{2\,.\,(2m-1)(2m+1)}\right\}\gamma\tau\cos\{(\tau+1)\lambda-\alpha\}$$

$$+\left\{(m-\tau+1)\frac{(m+\tau)(m+\tau+1)\,\psi_m{}^{\tau-1}-(m-\tau+2)(m-\tau+3)\,\psi_{m+2}{}^{\tau-1}}{2\,(2m+1)(2m+3)}\right.$$

$$\left.+(m+\tau)\frac{(m+\tau-1)(m+\tau-2)\,\psi_{m-2}{}^{\tau-1}-(m-\tau)(m-\tau+1)\,\psi_m{}^{\tau-1}}{2\,.\,(2m-1)(2m+1)}\right\}$$

$$\gamma\tau\cos\{(\tau-1)\lambda-\alpha\} \quad . \quad . \quad (37\ bis).$$

As, omitting constant factors, ψ_n^σ is equal to Q_n^σ, the tesseral function of type σ and degree n, we may now write

$$\rho\cos\theta\frac{d\psi}{d\lambda} = \Sigma f_n^\sigma \text{Q}_n^\sigma,$$

and tabulate those values of f_n^σ which are not equal to zero. The following table is constructed in this way.

VALUES of f_n^σ.

n ; σ =	$\tau-1$	τ	$\tau+1$
$m-2$	$\frac{(m+\tau)(m+\tau-1)(m+\tau-2)}{2(2m-1)(2m+1)}\tau\gamma$	$\frac{(m+\tau)(m+\tau-1)}{(2m-1)(2m+1)}\tau\gamma'$	$-\frac{m+\tau}{2(2m-1)(2m+1)}\tau\gamma$
$m-1$	—	$\frac{m+\tau}{2m+1}\tau$	—
m	$\frac{(m+\tau)(m-\tau+1)}{2(2m-1)(2m+3)}\tau(2\tau-1)\gamma$	$\frac{2(m^2-\tau^2)+(2m-1)}{(2m-1)(2m+3)}\tau\gamma$	$\frac{2\tau-1}{2(2m-1)(2m+3)}\tau\gamma$
$m+1$	—	$\frac{(m-\tau+1)}{2m+1}\tau$	—
$m+2$	$-\frac{(m-\tau+2)(m-\tau+3)}{2(2m+1)(2m+3)}\tau\gamma$	$\frac{(m-\tau+1)(m-\tau+2)}{(2m+1)(2m+3)}$	$\frac{(m-\tau+1)}{2(2m+1)(2m+2)}\tau\gamma$

The transformation of $-\rho \sin\theta \frac{dS}{d\theta}$ presents no difficulties. If S be expressed in its series according to (30) and the terms re-arranged, the result of the operation is of the form $\Sigma r_n^\sigma Q_n^\sigma$, where r_n^σ is made up of three parts. Independently of γ and γ', we have

$$\frac{(n+2)(n+\sigma+1)}{2n+3}\kappa^\sigma_{n+1}-\frac{(n-1)(n-\sigma)}{2n-1}\kappa^\sigma_{n-1} \quad . \quad . \quad . \quad . \quad . \quad (38).$$

We have further, multiplied by γ',

$$\frac{n+\sigma+1}{2n+3}\left[\frac{(n+3)(n+\sigma+2)}{2n+5}\kappa^\sigma_{n+2}-\frac{n(n-\sigma+1)}{2n+1}\kappa_n^\sigma\right]$$
$$+\frac{n-\sigma}{2n-1}\left[\frac{(n+1)(n+\sigma)}{2n+1}\kappa_n^\sigma-\frac{(n-2)(n-\sigma-1)}{2n-3}\kappa^\sigma_{n-2}\right] \quad . \quad . \quad (39),$$

and finally, multiplied by γ,

$$\frac{(n+\sigma+1)(n+\sigma+2)}{2\,.\,2n+3}\left[\frac{(n+3)(n+\sigma+3)}{2n+5}\kappa_{n+2}^{\sigma+1}-\frac{n(n-\sigma)}{2n+1}\kappa_n^{\sigma+1}\right]$$
$$-\frac{(n-\sigma)(n-\sigma-1)}{2(2n-1)}\left[\frac{(n+1)(n+\sigma+1)}{2n+1}\kappa_n^{\sigma+1}-\frac{(n-2)(n-\sigma-2)}{2n-3}\kappa_{n-2}^{\sigma+1}\right]$$
$$-\frac{1}{2\,.\,2n+3}\left[\frac{(n+3)(n+\sigma+1)}{2n+5}\kappa_{n+2}^{\sigma-1}-\frac{n(n-\sigma+2)}{2n+1}\kappa_n^{\sigma-1}\right]$$
$$+\frac{1}{2(2n-1)}\left[\frac{(n+1)(n+\sigma-1)}{2n+1}\kappa_n^{\sigma-1}-\frac{(n-2)(n-\sigma)}{2n-3}\kappa_{n-2}^{\sigma-1}\right] \quad . \quad . \quad . \quad . \quad (40).$$

By means of the values of κ already tabulated, the factors r may be calculated.

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R is expressible in the form

$$R = \sum_{\sigma=0}^{\sigma=\infty} \{p_n^\sigma \sin(\sigma\lambda-\alpha)+q_n^\sigma \sin(\sigma\lambda+\alpha)\} Q_n^\sigma \ldots,$$

or, admitting negative values of σ, more conveniently by

$$R = \sum_{\sigma=-\infty}^{\sigma=+\infty} p_n^\sigma Q_n^\sigma \sin(\sigma\lambda-\alpha).$$

In the ultimate result we return to the q coefficients through the relation

$$p_n^{-\sigma} = (-1)^{\sigma+1}\frac{(n+\sigma)\,!}{(n-\sigma)\,!}q_n^\sigma.$$

Equating the factors of $Q_n^\sigma \cos\sigma(\lambda-\alpha)$ in (14), we now find

$$\sigma p_n^\sigma = f_n^\sigma + r_n^\sigma.$$

The calculation of r_n^σ in its present form involves the summation of the expressions (38), (39), and (40), the κ factors being substituted out of the tables previously given. The somewhat troublesome labour involved in this process may almost entirely be avoided by a transformation of expression (38). Substituting A_n^σ from (34) into (33), we find

$$n\,.\,n+1\,.\,\kappa_n^\sigma = E_n^\sigma - B_n^\sigma\gamma' - C_n^\sigma\gamma,$$

and by means of this equation, when $n+1$ and $n-1$ are respectively substituted for m, we obtain

$$\frac{(n+2)(n+\sigma+1)}{2n+3}\kappa^\sigma_{n+1} - \frac{(n-1)(n-\sigma)}{2n-1}\kappa^\sigma_{n-1} = e_n^\sigma + (B^\sigma_{n-1}-B^\sigma_{n+1})\gamma' + (C^\sigma_{n-1}+C^\sigma_{n+1})\gamma \quad (41),$$

where

$$e_n^\sigma = \frac{n+\sigma+1}{(n+1)(2n+3)}E^\sigma_{n+1} - \frac{(n-\sigma)}{n(2n-1)}E^\sigma_{n-1}.$$

If the right-hand side of (41) replaces (38), and is added to (39) and (40), the whole expression reduces to

$$r_n^\sigma = e_n^\sigma + \frac{1}{n\,.\,n+1}[\tfrac{1}{2}\sigma\gamma\{\kappa_n^{\sigma-1}+(n-\sigma)(n+\sigma+1)\kappa_n^{\ +1}\} - \sigma^2\gamma'\kappa_n^\sigma].$$

We have, therefore, the following very convenient expression which allows us to calculate the p coefficients from the previously established values of κ :—

$$p_n^\sigma = \frac{1}{\sigma}(e_n^\sigma+f_n^\sigma) + \frac{1}{n\,.\,n+1}[\tfrac{1}{2}\gamma\{\kappa_n^{\sigma-1}+(n-\sigma)(n+\sigma+1)\kappa_n^{\sigma+1}\} - \sigma\gamma'\kappa_n^\sigma] \quad (42).$$

The first term on the right-hand side is zero, except for the cases where the type σ

does not differ by more than 1 and the degree n by not more than 2 from the type and degree of the original velocity potential.

For the calculation of these special cases the previous investigation furnishes the necessary formulæ. Confining ourselves to the velocity potentials ψ_1^1 and ψ_2^2, we find the requisite numbers collected in the following table :—

VALUES of $e_n^\sigma + f_n^\sigma$.

Velocity potential $= \psi_1^1$.			Velocity potential $= \psi_2^2$.			
$\sigma =$	1	2	$\sigma =$	1	2	3
$n = 1$	$\frac{2}{5}\gamma'$	—	$n = 2$	$\frac{1}{7}\gamma$	$\frac{10}{21}\gamma'$	—
2	$\frac{1}{6}$	—	3	—	$\frac{4}{15}$	—
3	$\frac{2}{45}\gamma'$	$\frac{1}{45}\gamma$	4	$-\frac{3}{70}\gamma$	$\frac{2}{35}\gamma'$	$\frac{3}{140}\gamma'$

Equation (42) holds also for negative values of σ, but when σ is smaller than -2, it is more convenient to calculate the q coefficients from the values of μ_n^σ already given. We find for this case

$$q_n^\sigma = \frac{1}{n\,.\,n+1}\left[\tfrac{1}{2}\gamma\left\{\mu_n^{\sigma-1} + (n+\sigma+1)(n-\sigma)\,\mu_n^{\sigma+1}\right\} - \sigma\gamma'\mu_n^\sigma\right],$$

which, as may be expected, is identical with the equation connecting p and κ.

For $\sigma = 1$, (42) gives

$$p_n^{-1} = \frac{1}{n\,.\,n+1}\left[\tfrac{1}{2}\gamma\left\{\kappa_n^{-2} + n(n+1)\,\kappa_n^0\right\} + \gamma'\kappa_n^{-1}\right]$$

$$= \tfrac{1}{2}\gamma\left\{(n-1)(n+2)\,\mu_n^2 + \kappa_n^0\right\} - \gamma'\mu'_n,$$

and hence

$$q'_n = \frac{1}{n\,.\,n+1}\left[\tfrac{1}{2}\gamma\left\{(n-1)(n+2)\,\mu_n^2 + \kappa_n^0\right\} - \gamma'\mu'_n\right]. \quad . \quad . \quad . \quad . \quad (43).$$

Our equations are not valid for the case $\sigma = 0$, because they depend on a division by σ. The first of equations (14) from which we started containing R only in the form $d\mathrm{R}/d\lambda$ is obviously unsuitable to determine those parts of R which are independent of λ; we must, therefore, have recourse to the second equation

$$\frac{d\mathrm{R}}{d\theta} = \rho\cos\theta\,\frac{d\psi}{d\theta} + \rho\,\frac{d\mathrm{S}}{\sin\theta\,d\lambda}\,.$$

Such products as $\rho \dfrac{d\psi}{d\lambda}$ and $\rho \dfrac{dS}{d\lambda}$ may give rise to terms independent of λ only through products of the form $\cos\lambda\cos(\lambda-\alpha)$, $\cos\lambda\cos(\lambda+\alpha)$ or $\cos\lambda\sin(\lambda-\alpha)$, $\cos\lambda\sin(\lambda+\alpha)$.

We need therefore only consider that part of ρ which involves $\cos\lambda$ and terms of the first type in ψ and S. Selecting the two terms $\{\kappa'_n\cos(\lambda-\alpha)+\mu'_2\cos(\lambda+\alpha)\}\,Q'_n$ in the expression for S and rejecting all terms not containing λ, we are left with

$$\frac{\rho\, dS}{\sin\theta\, d\lambda} = \tfrac{1}{2}\gamma\,(\mu'_n-\kappa'_n)\,Q'_n \sin\alpha.$$

Similarly

$$\rho\cos\theta\frac{d\psi}{d\theta} = -\tfrac{1}{2}\sin\theta\cos\theta\frac{dQ'_m}{d\theta}\sin\alpha.$$

By means of the formulæ of transformation previously given we find

$$-\sin\theta\cos\theta\frac{dQ'_m}{d\theta} = \frac{m\,.\,(m+1)^2}{(2m+1)(2m-1)}Q'_{m-2}+\frac{m^2+m-3}{(2m-1)(2m+3)}Q'_m-\frac{m^2(m+1)}{(2m+1)(2m+3)}Q'_{m+2}.$$

In the special case worked out in the previous pages in which $m = 1$,

$$-\tfrac{1}{2}\sin\theta\cos\theta\frac{dQ'_1}{d\theta} = -\tfrac{1}{10}Q'_1-\tfrac{1}{15}Q'_3.$$

The velocity potential $\psi_2^{\,2}$ contributes, as far as the term $\rho\cos\theta\dfrac{d\psi}{d\theta}$ is concerned, nothing to the zonal harmonics in R.

We have therefore in the second case, and whenever the velocity potential does not contain a term of the first type,

$$\frac{dR}{d\theta} = \tfrac{1}{2}\gamma\,(\kappa'_n-\mu'_n)\,Q'_n\sin\alpha.$$

If R, as far as its zonal harmonics are concerned, is written in the form $\Sigma p_n^{\,0}P_n\sin\alpha$, we find, as $-\dfrac{dP_n}{d\theta} = \sin\theta\dfrac{dP_n}{d\mu} = Q'_n$,

$$p_n^{\,0} = \tfrac{1}{2}\gamma\,(\kappa'_n-\mu'_n) \quad \ldots\ldots\ldots\ldots \quad (44).$$

We should have obtained the same value if we had applied the general equation to this case. When $\tau = 1$, we have to consider the terms depending on $\dfrac{d\psi}{d\theta}$, and must therefore write

$$p_1^{\,0} = \{-\tfrac{1}{10}+\tfrac{1}{2}(\kappa'_1-\mu'_1)\}\,\gamma,$$

$$p_3^{\,0} = \{-\tfrac{1}{15}+\tfrac{1}{2}(\kappa'_3-\mu'_3)\}\,\gamma.$$

For all other values of n equation (44) applies.

The tables which follow give the calculated values of p_n^σ and q_n^σ, and therefore solve the problem as far as terms of the fourth order in γ and γ'. It will be noticed that for unity, which is the highest admissible value of γ and γ', all the factors involving higher powers than the first are so small that their effect falls much below anything that observation is capable of showing. Hence the approximate calculation given in Part I. is sufficient for all practical purposes.

Restoring constant factors, we may summarise the result of the previous investigation as follows :—

1. *Notation.*

Q_n^σ denotes the tesseral function $\sin^\sigma\theta\, d^\sigma P_n/d\mu^\sigma$, where P_n is the zonal harmonic of degree n, and θ the colatitude.

C measured upwards denotes the vertical magnetic force of the earth's permanent field at the geographical pole.

e is the thickness of the conducting atmospheric shell.

ρ is the conductivity, which is supposed to be variable and depending on θ and the local time $\lambda+t$, according to the relation $\rho = \rho_0\,[1+\gamma'\cos\theta+\gamma\sin\theta\cos(\lambda+t)]$, where γ and γ' are constant.

$A_n^\sigma Q_n^\sigma \cos\{\sigma(\lambda+t)-\alpha\}$ is the velocity potential of the flow of air.

2. *Conclusion.*

The current function R of electric flow induced under the action of the vertical force $C\cos\theta$ in the oscillating shell of air is then expressed as a sum

$$R = A_n^\sigma C e\rho_0\left[\sum_{\sigma=0}^{\sigma=\infty} p_n^\sigma Q_n^\sigma \sin\{\sigma(\lambda+t)-\alpha\} + \sum_{\sigma=1}^{\sigma=\infty} q_n^\sigma Q_n^\sigma \sin\{\sigma(\lambda+t)+\alpha\}\right].$$

In order to obtain the magnetic potential of the variation caused by the flow of air, a factor $-4\pi(n+1)/(2n+1)$ has to be applied.

The factors p_n^σ and q_n^σ are given in the tables (including terms of the fourth order of γ and γ') for the velocity potentials

$$A'_1Q' = A'_1\sin\theta\cos\{(\lambda+t)-\alpha\}$$

and

$$A_2^2Q_2^2 = 3A_2^2\sin^2\theta\cos\{2(\lambda+t)-\alpha\}.$$

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I. VELOCITY Potential: $A_1 \sin\theta \cos\{(\lambda + t) - \alpha\}$.

$$p_1{}^0 = \frac{3}{20}\gamma + \frac{1}{80}\gamma^3 - \frac{1}{80}\gamma\gamma'^2.$$

$$p_2{}^0 = \frac{1}{72}\gamma\gamma' + \frac{89}{18144}\gamma\gamma'^3 - \frac{25}{18144}\gamma'\gamma^3.$$

$$p_3{}^0 = -\frac{1}{15}\gamma - \frac{1}{1080}\gamma^3 - \frac{1}{270}\gamma'^2\gamma.$$

$$p_4{}^0 = \frac{1}{3360}\gamma'\gamma^3 + \frac{1}{840}\gamma'^3\gamma.$$

$$p_1{}^1 = \frac{3}{20}\gamma' + \frac{1}{80}\gamma'^3.$$

$$p_2{}^1 = \frac{1}{6} - \frac{1}{216}\gamma'^2 - \frac{5}{432}\gamma^2 - \frac{89}{54432}\gamma'^4 - \frac{31}{15552}\gamma^4 - \frac{89}{36288}\gamma^2\gamma'^2.$$

$$p_3{}^1 = \frac{2}{45}\gamma' + \frac{1}{648}\gamma'\gamma^2 + \frac{1}{1620}\gamma'^3.$$

$$p_4{}^1 = -\frac{1}{4200}\gamma^2\gamma'^2 - \frac{1}{8400}\gamma'^4 + \frac{1}{9600}\gamma^4.$$

$$p_2{}^2 = \frac{1}{144}\gamma\gamma' + \frac{1}{1134}\gamma'\gamma^3 + \frac{139}{108864}\gamma\gamma'^3.$$

$$p_3{}^2 = \frac{1}{90}\gamma + \frac{1}{2592}\gamma^3 - \frac{1}{3240}\gamma\gamma'^2.$$

$$p_4{}^2 = -\frac{23}{201600}\gamma'\gamma^3 + \frac{1}{50400}\gamma\gamma'^3.$$

$$p_3{}^3 = -\frac{1}{4320}\gamma'\gamma^2.$$

$$p_4{}^3 = \frac{1}{40320}\gamma^2\gamma'^2 - \frac{1}{80640}\gamma^4.$$

$$p_4{}^4 = \frac{1}{134400}\gamma'\gamma^3.$$

$$q_1{}^1 = \frac{1}{80}\gamma^2\gamma'.$$

$$q_2{}^1 = -\frac{1}{144}\gamma^2 - \frac{89}{36288}\gamma^2\gamma'^2 - \frac{13}{36288}\gamma^4.$$

$$q_3{}^1 = \frac{1}{720}\gamma^2\gamma'.$$

$$q_4{}^1 = -\frac{13}{33600}\gamma^2\gamma'^2 + \frac{19}{67200}\gamma^4.$$

$$q_2{}^2 = -\frac{19}{36288}\gamma'\gamma^3.$$

$$q_3{}^2 = \frac{1}{4320}\gamma^3.$$

$$q_4{}^2 = -\frac{9}{67200}\gamma'\gamma^3.$$

$$q_3{}^3 = 0.$$

$$q_4{}^3 = \frac{1}{134400}\gamma^4.$$

II. Velocity Potential: $3A_2^2 \sin^2\theta \cos\{2(\lambda+t)-\alpha\}$.

$$p_1^0 = -\frac{1}{105}\gamma^4.$$

$$p_2^0 = 0.$$

$$p_3^0 = -\frac{1}{36}\gamma^2 - \frac{23}{12960}\gamma^4 - \frac{133}{25920}\gamma^2\gamma'^2.$$

$$p_4^0 = \frac{1}{80}\gamma'\gamma^2.$$

$$p_5^0 = -\frac{1}{175}\gamma^2\gamma'^2 + \frac{1}{1575}\gamma^4.$$

$$p_1^1 = -\frac{1}{210}\gamma'\gamma^3.$$

$$p_2^1 = \frac{16}{63}\gamma - \frac{4}{567}\gamma\gamma'^2 + \frac{5}{567}\gamma^3.$$

$$p_3^1 = \frac{1}{72}\gamma\gamma' - \frac{41}{51840}\gamma'\gamma^3 + \frac{191}{51840}\gamma\gamma'^3.$$

$$p_4^1 = -\frac{3}{70}\gamma + \frac{1}{5600}\gamma^3 - \frac{19}{5600}\gamma\gamma'^2.$$

$$p_5^1 = \frac{23}{23625}\gamma\gamma'^3 - \frac{11}{31500}\gamma'\gamma^3.$$

$$p_2^2 = \frac{8}{63}\gamma' - \frac{2}{567}\gamma^2\gamma' + \frac{4}{567}\gamma'^3.$$

$$p_3^2 = \frac{2}{15} - \frac{1}{270}\gamma'^2 - \frac{1}{270}\gamma^2 - \frac{1}{38880}\gamma'^2\gamma^2 - \frac{53}{62208}\gamma^4 - \frac{11}{9720}\gamma'^4.$$

$$p_4^2 = \frac{1}{35}\gamma' + \frac{1}{5600}\gamma'\gamma^2 + \frac{1}{2100}\gamma'^3.$$

$$p_5^2 = \frac{1}{15750}\gamma^2\gamma'^2 + \frac{1}{63000}\gamma^4 - \frac{2}{23625}\gamma'^4.$$

$$p_3^3 = \frac{1}{432}\gamma\gamma' - \frac{41}{311040}\gamma\gamma'^3 + \frac{133}{311040}\gamma'\gamma^3.$$

$$p_4^3 = \frac{1}{140}\gamma - \frac{1}{33600}\gamma\gamma'^2 + \frac{1}{8400}\gamma^3.$$

$$p_5^3 = -\frac{1}{94500}\gamma\gamma'^3 - \frac{1}{42000}\gamma^3\gamma'.$$

$$p_4^4 = -\frac{1}{13440}\gamma'\gamma^2.$$

$$p_5^4 = +\frac{1}{189000}\gamma^2\gamma'^2 - \frac{1}{283500}\gamma^4.$$

$$p_5^5 = \frac{1}{453600}\gamma'\gamma^3.$$

$$q_1^1 = -\frac{1}{70}\gamma^3\gamma'.$$

$$q_2^1 = \frac{1}{189}\gamma^3.$$

II. Velocity Potential: $3A_2^2 \sin^2\theta \cos\{2(\lambda+t)-\alpha\}$ (continued).

$$q_3^1 = -\frac{29}{25920}\gamma^3\gamma'.$$

$$q_4^1 = -\frac{1}{1120}\gamma^3.$$

$$q_5^1 = \frac{11}{18900}\gamma'\gamma^3.$$

$$q_2^2 = 0.$$

$$q_3^2 = -\frac{29}{103680}\gamma^4.$$

$$q_4^2 = 0.$$

$$q_5^2 = \frac{1}{37800}\gamma^4.$$